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Hamilton decompositions of two-ended infinite graphs

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Abstract

A Cayley graph $Cay(\Gamma, S)$ is a graph with vertex set the elements of the group Γ, and edge set $\{\{u, v\} \mid v = u + s, s \in S \subset \Gamma\}$. A Hamilton cycle is a closed path which visits every vertex in a graph exactly once, and a Hamilton decomposition of a graph is a partition of its edge-set into Hamilton cycles. It has been conjectured by Alspach that every connected $2k$-regular Cayley graph of a finite abelian group has a Hamilton decomposition. Another conjecture from Alspach and Rosenfeld says that if G is a 3-connected, 3-regular graph with a Hamilton cycle, then the Cartesian product $G \Box K_2$ is Hamilton decomposable. Bermond also conjectures that if two graphs are Hamilton decomposable, then their Cartesian product is also Hamilton decomposable. These three conjectures remain open, and we examine their infinite generalisations. Specifically, we study two-ended infinite abelian groups, and the Cartesian product $G \Box P_\infty$, where G is an even-regular finite graph and P_∞ is a two-way infinite path. The notion of a Hamilton cycle can be generalised to infinite graphs as a spanning two-way infinite path. However, if the graph has two ends, some graphs contain an end-separating edge-cut which differs in parity from the number of paths in the decomposition, precluding the existence of a decomposition into spanning two-way infinite paths. The question, then, is whether this is the only obstruction to a Hamilton decomposition.

We show that if G is 2, 4, or 6-regular, and has a Hamilton decomposition, then $G \Box P_\infty$ has a Hamilton decomposition if it avoids the obstruction. We show that a two-ended Cayley graph with only one non-torsion generating element can be viewed as a graph of the form $G \Box P_\infty$, and thus complete the case of the generalisation of Alspach's conjecture for two-ended, 6-regular Cayley graphs of infinite abelian groups with only one non-torsion generating element. We further investigate the existence of a Hamilton decomposition of $G \Box P_\infty$ if G has a Hamilton cycle, but not a Hamilton decomposition, and construct a backtracking algorithm to test the generalisation of Alspach's conjecture and our techniques, in the cases which remain unproven. An infinite family of positive solutions is provided in this unproven case, as well as examples which do not fit the techniques used.

To my grandparents,
each of you has played a unique and immeasurable role in making me who I am today.

Contents

Chapter 1

Introduction

1.0.1 Standard notation

Much of the notation in this thesis is standard in graph theory texts, but to avoid ambiguity, the notation used throughout the thesis is covered here. Let C_n be the cycle on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$, edge set $\{\{v_i, v_{i+1}\} \mid i = 1, \ldots, n-1\} \cup \{\{v_n, v_1\}\}$, denoted by $(v_1, \ldots, v_n)$. K_n the complete graph on n vertices, and λK_n the complete multi-graph where there are λ edges between each pair of vertices. For two vertices x and y, the notation $x \sim y$ means x is adjacent to y. We call the degree of a vertex or a graph its valency. When it is clear, the edge $\{u, v\}$ will be written as uv. For two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, we define the union of two graphs $G \cup H$ as $(V_1 \cup V_2, E_1 \cup E_2)$, and note that it is not the disjoint union. For a graph $G = (V, E)$ and a vertex subset $V' \subset V$ or edge subset $E' \subset E$, we define $G[V']$ as the subgraph of G induced by the vertices in V', and $G[E']$ as the subgraph of G induced by the edges in E'. Let $P = [a_1, a_2, \ldots, a_n]$ be a path from vertex a_1 to a_n, with vertex set $V(P) = \{a_1, \ldots, a_n\}$ and edge set $a_i a_{i+1}$ for all $1 \leq i < n$, and let $\hat{P} = [a_n, \ldots, a_2, a_1]$ be the reversal of P.

1.1 Finite graphs

Let $G = (V, E)$ be a finite graph with vertex set V and edge set E. If, for some set of cycles $H_0, \ldots, H_k$ and a finite graph G, each cycle spans G, $E(H_0) \cup \cdots \cup E(H_k) = E(G)$, and $E(H_i) \cap E(H_j) = \emptyset$ for all $i \neq j$, then we say that $H_0, \ldots, H_k$ is a Hamilton decomposition of G, and we say G is Hamilton decomposable. It is clear that G must be even-regular and connected for such a decomposition to exist. It is also necessary that the graph be $2k$-edge connected. The foundational result in Hamilton decompositions is credited to Walecki by Lucas in 1891 [13], who gives a construction for a Hamilton decomposition of any complete graph of odd order [2]. Since then, Hamilton decompositions of a wide variety of graph families have been studied, and the problem of determining whether or not an arbitrary graph has a Hamilton decomposition is known to be NP-complete [31].

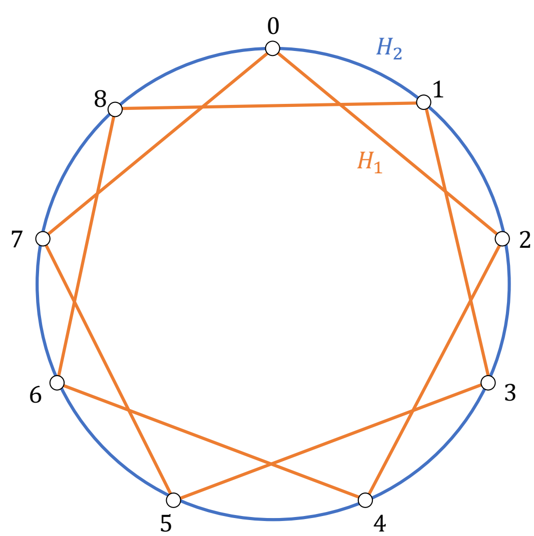


Figure 1.1: A graph with a Hamilton decomposition into two cycles.

1.1.1 Cayley graphs

One important family of graphs is Cayley graphs, due to their usefulness in representing and understanding group structures, and their symmetry. In particular, we study Cayley graphs of abelian groups.

Definition 1 (Cayley graph $Cay(\Gamma, S)$)**.** Let $(\Gamma, +)$ be a finitely generated abelian group. For an inverse-closed subset S of $\Gamma \setminus \{0\}$ ($s \in S \Rightarrow -s \in S$), let S^+ be a subset of S such that $S^+ \cup -S^+ = S$ and $S^+ \cap -S^+ = \emptyset$, where $-S = \{-s \mid s \in S\}$, and call S^+ a *positive* generating set. The Cayley graph $Cay(\Gamma, S)$ is defined to be the graph of Γ generated by S, where $Cay(\Gamma, S)$ has vertex set Γ and edge-set $\{\{x, x \pm g\} \mid x \in \Gamma, g \in S^+\}$.

Note that $Cay(\Gamma, S)$ is connected if and only if S generates Γ. If $|g| = 2$ for some $g \in S$, then the edges $\{x, x+g\}$ and $\{x, x-g\}$ will have the same endpoints, and in this case we allow $Cay(\Gamma, S)$ to be a multi-graph, in keeping with the notation in [7]. In doing so, the valency of $Cay(\Gamma, S)$ is always $2|S^+| = |S|$. It is also easy to confirm that every Cayley graph is vertex-transitive, as the adjacency of a vertex $x \in V(Cay(\Gamma, S))$ with its neighbours $\{x + s, s \in S\}$ is preserved by the action of the group element $y - x \in \Gamma$ which maps x to any other vertex $y \in V(Cay(\Gamma, S))$. In 1982, Marušič [28] proved that every connected Cayley graph on a finite abelian group has a Hamilton cycle (i.e. it is *Hamiltonian*). For a family of graphs that all contain at least *one* Hamilton cycle, the natural follow-up question is which graphs can be *decomposed* into Hamilton cycles. For a Hamilton decomposition into k cycles to exist, a graph must be even-regular, and $2k$-edge connected. A Cayley graph is $2k$-edge connected as long as it is connected. The conjecture of Alspach is that $2k$-regular and connected are the only restrictions.

Conjecture 1 (Alspach 1984 [1])**.** *If Γ is a finite abelian group, then the Cayley graph $Cay(\Gamma, S)$ has a Hamilton decomposition into k cycles if it is connected, and $2k$-regular for some integer k.*

In 1989, this conjecture was proven in the 4-regular case by Bermond, Favaron and Maheo [7], but the 6-regular and above case is still open in general, with some partial results due to Liu [25, 26, 27], Westlund [37], Dean [11, 12], and Fan, Lick and Liu [16]. If S is a minimal generating set for Γ, then Liu [26, 27] proved that $Cay(\Gamma, S)$ always has a Hamilton decomposition.

1.1.2 Cartesian products

Another important family of graphs is the Cartesian product of two graphs. (See Figure 1.2 for an example.)

Definition 2 (Cartesian product of graphs)**.** Let G and H be graphs. The Cartesian graph product $G \square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if and only if:

- $x = x'$ and y is adjacent to y' in H, or
- $y = y'$ and x is adjacent to x' in G.

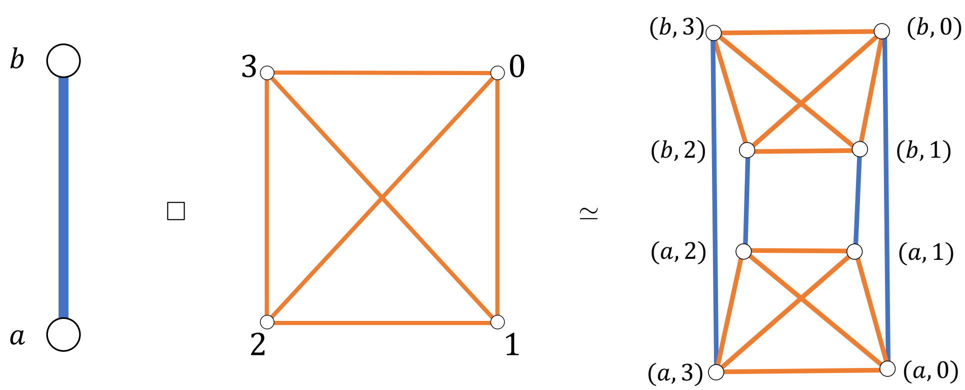


Figure 1.2: K_2, K_4, and $K_2 \square K_4$.

Note that if G and H are connected and r_g-regular and r_h-regular respectively, then $G \square H$ will be connected and $(r_g + r_h)$-regular. The first result in Hamilton decompositions of Cartesian products is due to Kotzig in 1973, who proved that for two cycles C_n and C_m, $C_n \Box C_m$ is Hamilton decomposable. (A proof of Kotzig's result is presented in Foregger's 1978 paper [17].) Various other families of Cartesian products have been studied since, with perhaps the most general conjecture proposed by Bermond in [6].

Conjecture 2 (Bermond [6])**.** *If G_1 and G_2 are finite and have Hamilton decompositions, then $G_1 \Box G_2$ also has a Hamilton decomposition.*

Foregger's 1978 paper [17] also presents a proof of Kotzig's conjecture that $C_r \Box C_s \Box C_t$ is Hamilton decomposable. In 1982, Aubert and Schneider [5] proved that if C is a cycle and G is a 4-regular, Hamilton decomposable graph, then $C \Box G$ is decomposable into three Hamilton cycles. These results also have two corollaries which are proven by Alspach, Bermond and Sotteau [3]. The first is that if G_1 has a decomposition into n Hamilton cycles and G_2 has a decomposition into m Hamilton cycles, where $n \leq m \leq 2n$, then $G_1 \Box G_2$ can be decomposed into $n+m$ Hamilton cycles. The second corollary, which is derived by induction from the first, is that for a collection of cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_n}$, $C_{i_1} \Box C_{i_2} \Box \ldots \Box C_{i_n}$ has a Hamilton decomposition. The most significant result towards proving Bermond's Conjecture was achieved by Stong [36], who proved that for graphs G_1 and G_2 which are decomposable into n and m Hamilton cycles respectively, with $n \leq m$, $G_1 \Box G_2$ is Hamilton decomposable under any of the following conditions:

1. $m \leq 3n$,
2. $n \geq 3$,
3. $|G_1|$ even, or
4. $|G_2| \geq 6 \lceil m/n \rceil - 3$.

One can also study *prisms*, any graph of the form $G \Box K_2$. $G \Box K_2$ is even-regular and connected as long as G is odd-regular and connected. Alspach and Rosenfeld [4] proved that if G is 3-regular and has a perfect 1-factorisation, then $G \Box K_2$ has a Hamilton decomposition, where a perfect 1-factorisation is a decomposition into 1-factors such that the union of any pair of 1-factors is a Hamilton cycle. They then suggested the following conditions for a Hamilton decomposition to exist.

Conjecture 3 (Alspach and Rosenfeld [4])**.** *If G is a 3-connected, 3-regular graph having a Hamilton cycle, $G \Box K_2$ has a Hamilton decomposition.*

They also showed that this condition on G is not necessary for $G \Box K_2$ to be Hamilton decomposable, by showing that for the Petersen graph P, $P \Box K_2$ has a Hamilton decomposition, even though P does not have a Hamilton cycle. A review of the current state of the conjecture was written by Rosenfeld and Xiang in 2015 [32]. It has been proven for graphs of the form $C_n \Box K_2$, 3-connected cubic bipartite planar graphs, and duals of kleetopes (cubic graphs that are obtained by starting from K_4 and repeatedly inflating vertices to triangles), but the conjecture remains open in general.

1.2 Infinite graphs

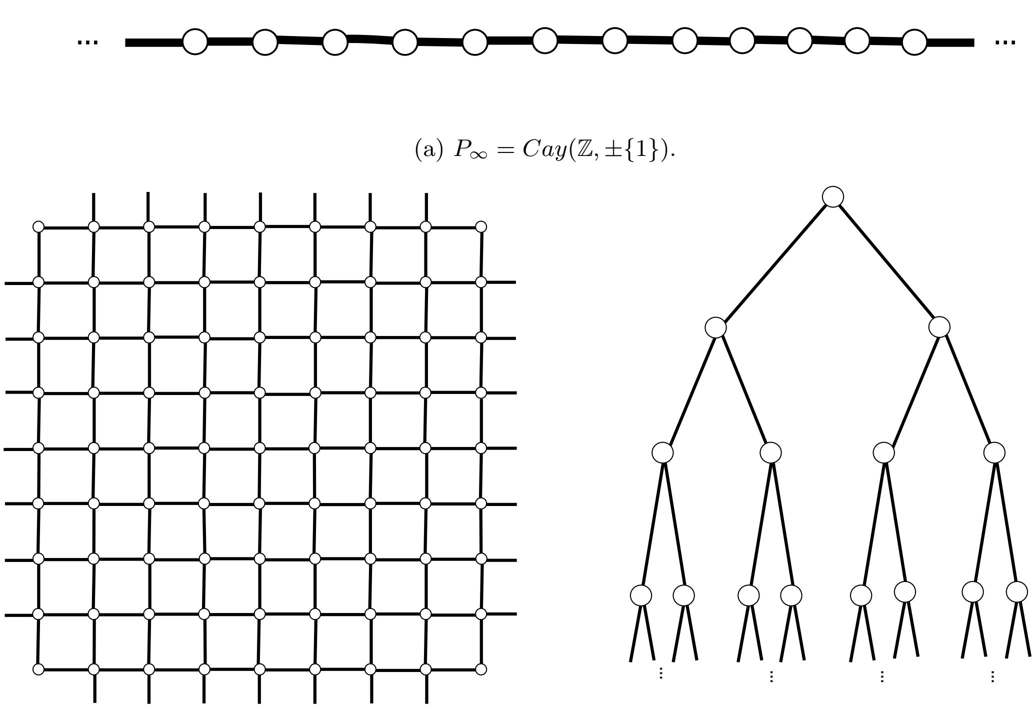
Given the conjectures and results for finite graphs discussed above, it is natural to ask how these generalise to infinite graphs. However, before investigating this in detail, one must first establish what it means to decompose an infinite graph into Hamilton cycles. In a finite graph, a Hamilton cycle provides two finite, internally vertex-disjoint paths between any two vertices. However, in a graph with an infinite number of vertices, at least one of these two paths would be infinite, otherwise the cycle would not be spanning. Instead, the most natural infinite generalisation of a Hamilton cycle borrows its property as a connected, spanning, 2-regular subgraph. In an infinite graph, a connected, spanning, 2-regular subgraph is a two-way infinite spanning path, also known as a Hamilton double-ray.

Definition 3 (Rays and double-rays)**.** A *ray* is defined as a one-way infinite path starting at vertex v_0, notated as $[v_0, v_1, v_2, \ldots]$. For two rays $[v_0, v_1, v_2, \ldots]$ and $[v_0, v_{-1}, v_{-2}, \ldots]$ that are vertex-disjoint except for the common starting vertex v_0, we define a *double-ray* as the two-way infinite path $[\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots]$. If a double-ray is spanning, then it is called a Hamilton double-ray.

Using double-rays one can then formulate an infinite generalisation of a Hamilton decomposition. If, for some set of spanning double-rays $Q_0, \ldots, Q_k$ in an infinite graph G_∞, $E(Q_0) \cup \cdots \cup E(Q_k) = E(G_\infty)$, and $E(Q_i) \cap E(Q_j) = \emptyset$ for all $i \neq j$, then we say that $Q_0, \ldots, Q_k$ is a Hamilton decomposition of G_∞ into double-rays, and we say G_∞ is Hamilton decomposable. One might hope that some of the conjectures for finite graphs in Section 1.1 generalise naturally to infinite graphs, but natural obstructions arise in the structure of certain infinite graphs which put an end to such idealistic desires. To understand these obstructions, we must first establish a notion of *ends* of a graph.

1.2.1 Ends of graphs

To motivate the following section, consider the following three infinite graphs: the double-ray $P_\infty = Cay(\mathbb{Z}, \pm\{1\})$, the two-dimensional integer grid $G = Cay(\mathbb{Z}^2, \pm\{(0,1),(1,0)\})$ and T, the infinite complete binary tree. (See Figure 1.3) Intuitively, it is clear that all of these graphs have different infinite structures. Specifically, they all seem to have different ways of extending to infinity. One might differentiate these structures by asking how many *ends* each graph has, in other words, how many different infinite directions the graph has. This notion can be formalised with the following definition.



(a) $P_\infty = Cay(\mathbb{Z}, \pm\{1\})$.

(b) $G = Cay(\mathbb{Z}^2, \pm\{(0,1),(1,0)\})$.

(c) T, The infinite complete binary tree.

Figure 1.3: Three infinite graphs, each with a different topology.

Definition 4 (End of a graph)**.** An end of a graph G is an equivalence class of rays in G, where two rays R_0 and R_1 are considered equivalent if there is no finite set of vertices $S \subset V(G)$ which separates them into distinct connected components of $G \setminus S$.

This definition can be used to classify rigorously the topological difference we observe between P_∞, G and T. P_∞ has infinitely many rays, one extending in each direction from each vertex. Yet any ray R_0 which extends leftwards and any ray R_1 which extends rightwards can be separated by a set of vertices which includes all vertices between and including the start vertices of R_0 and R_1. Any two rays which extend in the same direction share an infinite number of vertices, so no finite vertex-set can separate them. Thus, P_∞

has precisely two ends. G has no more than one end as there is no finite set of vertices which separates the graph, and it has at least one because it is infinite. Thus, it has exactly one end. T has continuum many ends as any ray which starts at the root can be determined by a sequence of left-right branching decisions down from the root. Any ray which does not start at the root only differs from some ray that does by a finite number of edges, so the two rays are equivalent. This makes the equivalence classes of rays of T equivalent to the closed interval of real numbers $[0, 1]$, where $x \in [0, 1]$ is mapped to an equivalence class of rays by the unique binary expansion of the ray starting at the root, where 0 is a left turn and 1 a right turn. Each of these rays can be separated by a set of vertices which includes all their common vertices, which is equivalent to determining the point at which their binary expansions differ. Thus, T has continuum many ends.

1.2.2 Ends of groups

The notion of ends of graphs can be utilised to define the ends of groups via Cayley graphs. The number of ends of a group Γ is defined to be the number of ends of $Cay(\Gamma, S)$ for some generating set S. It is proven by Scott and Wall in [34] that the number of ends of $Cay(\Gamma, S)$ is not dependent on the choice of generating set S. We can therefore freely refer to the ends of groups and their Cayley graphs as structural invariants. We now use this definition to classify the structure of the finitely generated abelian groups which are one of the main objects of the thesis. To do so, we start by fixing a notation justified by the fundamental theorem of finitely generated abelian groups [19], which states that every finitely generated abelian group Γ is isomorphic to

$$\mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_t} \oplus \mathbb{Z}^n$$

where $n \geq 0$, $q_1, \ldots, q_t$ are powers of not necessarily distinct primes, and $\oplus$ denotes the standard direct sum of groups. Let $\Gamma_{\text{fin}} \simeq \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_t}$ be the largest finite subgroup of Γ, so that $\Gamma \simeq \Gamma_{\text{fin}} \oplus \mathbb{Z}^n$.

We then present the following claim, which classifies the ends of finitely generated abelian groups purely in terms of n.

Claim 1. *A finitely generated abelian group $\Gamma_{fin} \oplus \mathbb{Z}^n$ has:*

- *zero ends iff $n = 0$.*
- *two ends iff $n = 1$.*
- *one end iff $n \geq 2$.*

The proof will be delayed until after the presentation of a few key results from the literature, after which the claim falls out as a corollary. Any terms from the following results which are not defined in this thesis are as is standard in the literature.

Theorem 1 (Freudenthal [18] and Hopf [23])**.** *A finitely generated group has either zero, one, two or infinitely many ends.*

Theorem 2 (Freudenthal [18] and Hopf [23])**.** *A finitely generated group Γ has two ends iff Γ contains an infinite cyclic subgroup of finite index.*

The original proofs of Theorems 1 and 2 are due independently to both Freudenthal ([18] 6.14, 6.15, 6.16) and Hopf ([23] *Satz II & V*). A proof is available in English due to Meier ([29] Theorem 11.27 and Corollary 11.34).

Theorem 3 (Wall and Scott, [34] Lemma 5.6)**.** *If H is a subgroup of finite index in Γ, then H and Γ have the same number of ends.*

Theorem 4 (Stallings, [35] 0.1)**.** *Let Γ be a finitely generated, group where the only finite-order element is the identity. Then Γ has infinitely many ends iff Γ is a free product, $\Gamma \simeq H * K$, where neither H nor K is the trivial group.*

Corollary 1. *A finitely generated abelian group has zero, one, or two ends.*

Proof. For any abelian group $\Gamma_{\text{fin}} \oplus \mathbb{Z}^n$, $\mathbb{Z}^n$ is a subgroup of finite index, so by Theorem 3, $\Gamma_{\text{fin}} \oplus \mathbb{Z}^n$ and $\mathbb{Z}^n$ have the same number of ends. Thus, it suffices to show that an abelian group on $\mathbb{Z}^n$ cannot have an infinite number of ends. Assume for contradiction that $\mathbb{Z}^n$ has an infinite number of ends. $\mathbb{Z}^n$ is a finitely generated group where the identity is the only finite-order element, so by Theorem 4, $\mathbb{Z}^n$ is a non-trivial free product. However this is a contradiction, as $\mathbb{Z}^n$ is abelian, and a non-trivial free product is not. Thus, any abelian group has a finite number of ends. □

Proof of Claim 1. Let $\Gamma = \Gamma_{\text{fin}} \oplus \mathbb{Z}^n$. The zero ends case is trivial, as Γ is finite. Therefore, assume Γ is infinite, so $n \geq 1$. If $n = 1$, the subgroup $H = \langle (0, \ldots, 0, 1) \rangle \simeq \mathbb{Z}$ is infinite, cyclic, and $[\Gamma : H] = |\mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_t}|$ is finite. So by Theorem 2, Γ has two ends. If $n \geq 2$, the only finite index subgroup is $H = A \oplus \mathbb{Z}^n$ for A some subgroup of Γ_{fin}. H requires at least $n \geq 2$ generators, and thus is not cyclic, so it cannot have two ends. As it has at least one end, and not infinite ends, it therefore must have one end. □

Given that the focus of this thesis is on two-ended graphs, we can thus say that $n = 1$, and this justifies the notation which will be used for two-ended, finitely generated abelian groups throughout:

$$\forall u \in \Gamma, u = (x, y), x \in \Gamma_{\text{fin}}, y \in \mathbb{Z} \tag{1.1}$$

We note that $x \in \Gamma_{\text{fin}}$ could itself be represented by a tuple i.e. $\Gamma_{\text{fin}} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_3$, but we will continue to represent it with a single symbol, as the structure of Γ_{fin} is not typically of importance.

1.3 Alspach's conjecture generalised

The infinite generalisation of Alspach's conjecture for finitely generated abelian groups is incredibly natural, what happens when we remove the condition that the group is finite?

Question 1. *Is every even-regular, connected Cayley graph of a finitely generated abelian group Hamilton decomposable?*

As was done with finite graphs, it should first be verified that every Cayley graph of a finitely generated abelian group contains a Hamilton double-ray, and this was proven by Nash-Williams [30]. The case is settled in the positive if the valency of the graph is countably infinite, as Witte [39] proved that an infinite graph with infinite valency has a Hamilton decomposition if and only if it has infinite edge-connectivity and a Hamilton double-ray. The condition of infinite edge-connectivity is essential, as if the edge-connectivity is only finite, a bottleneck is created through which an infinite number of double-rays could not pass. Combined with Nash-Williams' result, a connected Cayley graph on a finitely generated abelian group is Hamilton decomposable if it has infinite valency. The question, then, is what can be said about locally finite graphs?

For locally finite graphs, it is important to distinguish between generating elements with finite order, and those with infinite order. We define a *torsion* element of a group to be one with finite order, and a torsion edge as an edge in a Cayley graph generated by a torsion element. Similarly, a *non-torsion* element of a group is one with infinite order, and likewise for a non-torsion edge.

1.3.1 Admissibility criteria for two-ended graphs

The following admissibility criteria were first established by Bryant, Herke, Maenhaut and Webb [10] in the case where $\Gamma \simeq \mathbb{Z}$, and extended to any two-ended graph by Erde, Lehner and Pitz [15]. The admissibility criteria are derived from the fact that, in a two-ended graph, each Hamilton double-ray must use an odd number of edges from every finite, end-separating, edge cut-set. If a double-ray were to use an even number of edges from such a cut-set, then it would only visit a finite number of vertices on one side of the cut, and thus would not be spanning, as both sides are infinite.

As in [10], the criteria are formulated as a statement about the generating elements. Let Γ be the underlying group, and S the generating set of $Cay(\Gamma, S)$. (See Figure 1.4 for an example.) As established in section 1.2.2, each $g_i \in \Gamma$ can be expressed as $g_i = (x_i, y_i)$, for some $x_i \in \Gamma_{\text{fin}}$, $y_i \in \mathbb{Z}$.

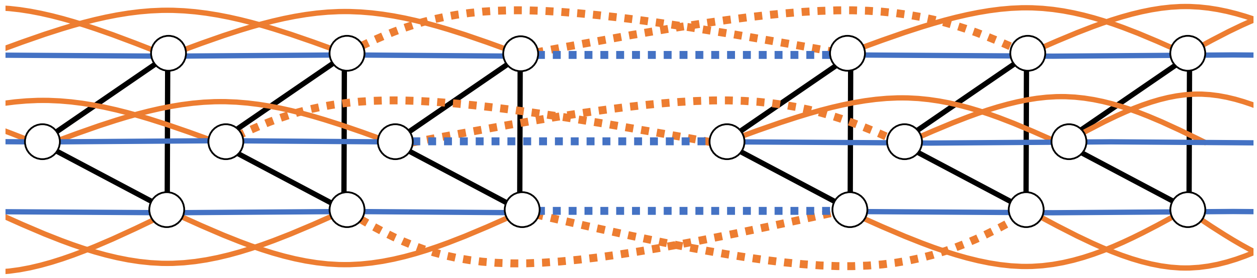


Figure 1.4: $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,0),(0,2)\}$ with an end-separating edge-cut identified by dotted edges.

Lemma 1. *If $Cay(\Gamma, S)$ is Hamilton-decomposable into k double-rays, where $\Gamma = \Gamma_{fin} \oplus \mathbb{Z}$, S is the disjoint union of S^+ and $-S^+$, and $S^+ = \{(x_i, y_i) \mid 1 \leq i \leq k\}$, then:*

1. *$Cay(\Gamma, S)$ is connected*
2. *$|\Gamma_{fin}| \sum_{i=1}^{k} y_i \equiv k \pmod 2$*

Proof. Consider the partition of Γ into $V_1 = \{(x,y) \mid y \leq 0\}$ and V_2 $\{(x,y) \mid y > 0\}$. The size of the edge cut-set $E = \{v_1v_2 \in E(Cay(\Gamma, S)) \mid v_1 \in V_1, v_2 \in V_2\}$ is only dependent on y_i (the infinite-order component) of each generating element, and the order of Γ_{fin}. The number of edges in E corresponding to a generating element (x_i, y_i) is exactly $|\Gamma_{\text{fin}}| y_i$, so $|E| = |\Gamma_{\text{fin}}| \sum_{i=1}^{k} y_i$. The valency of each vertex is $2k$, and so the Hamilton decomposition will consist of k double-rays, each of which must use an odd number of edges of E. So $|E| = |\Gamma_{\text{fin}}| \sum_{i=1}^{k} y_i \equiv k \pmod 2$. □

Note that the above Lemma presents two admissibility criteria, but we will often talk specifically about passing *the admissibility criterion*, that being the non-trivial condition that the parity of the edge cut-set is the same as the number of double-rays.

1.3.2 Existing results on the generalisation of Alspach's conjecture

In the one-ended case, Erde, Lehner and Pitz [15] showed that if S contains no torsion elements, $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}^n, S)$, $n \geq 2$ always has a Hamilton decomposition. If $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}^n, S)$ is 4-regular, then both generating elements must be non-torsion, otherwise S would not generate $\Gamma_{\text{fin}} \oplus \mathbb{Z}^n$, and $n = 2$ as two generators cannot generate $\mathbb{Z}^n$ if $n \geq 3$. In the two-ended, 4-regular case, Bryant, Herke, Maenhaut and Webb [10] show that if $Cay(\mathbb{Z}, S)$ passes the admissibility criteria, then it has a Hamilton decomposition. Erde and Lehner [14] show that if $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ is 4-regular and passes the admissibility criteria, then it has a Hamilton decomposition, as well as showing that $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ can be decomposed in the topologically similar notion of a Hamilton circle if it does not satisfy the admissibility criterion. These Hamilton circles come without the guarantee of a finite number of edges between any two vertices on the circle, and are only connected in a *compactification* of the graph, and are therefore not studied in this thesis. Together, these results solve the generalised conjecture for 4-regular graphs.

In the 6-regular case and higher, Erde, Lehner and Pitz's result for non-torsion generating elements in one-ended graphs applies, but is no longer a complete result for one-ended graphs, and Bryant, Herke, Maenhaut and Webb [10] derive partial results for $Cay(\mathbb{Z}, S)$, which are extended by Gentle in his Honours thesis [20]. No results are known in the 6-regular case for two-ended graphs with torsion elements. This is one of the main focuses of the thesis, and covered specifically in Chapter 4.

1.4 Cartesian product conjectures generalised

Unlike with generalising Alspach's Conjecture, it is less immediately clear how one would generalise Bermond's Conjecture for Cartesian products of Hamilton decomposable graphs, or Alspach and Rosenfeld's Conjecture for prisms of 3-connected cubic graphs with a Hamilton cycle. Perhaps the easiest generalisation of Bermond's Conjecture is to allow one, or both, of the graphs in the product to be infinite.

1.4.1 Bermond's Conjecture generalised

Conjecture 4. *If G and H have Hamilton decompositions into cycles or double-rays, then $G\square H$ has a Hamilton decomposition into cycles or double-rays.*

In the case of infinite-valency graphs, the result of Witte [39] that an infinite graph with infinite-valency has a Hamilton decomposition if and only if it has infinite edge-connectivity and a Hamilton double-ray could be applied. Without loss of generality, if we let H have infinite valency, then so does $G\square H$, and $G\square H$ is also infinitely edge-connected. Thus, it is only required to show that $G\square H$ has a Hamilton double-ray, but no result has been found in the literature to verify this.

In the case where G and H are both locally finite infinite graphs, Erde, Lehner and Pitz [15] show that if G and H both have Hamilton decompositions into double-rays, then $G\square H$ has a Hamilton decomposition too. The question remains open, however, in the case of the Cartesian product of a finite graph G with a locally finite, infinite graph H. In Chapter 2 we establish various results when H is the double-ray.

1.4.2 Alspach and Rosenfeld's Conjecture generalised

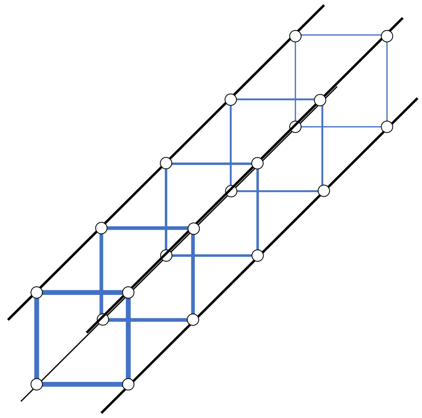
The conjecture of Alspach and Rosenfeld states that if G is a 3-connected cubic graph with a Hamilton cycle, then $G\square K_2$ is Hamilton decomposable. The infinite generalisation could be applied to either G or K_2. If we no-longer require G to be finite, one might ask what conditions are required on an infinite graph G such that if G is cubic, $G\square K_2$ is decomposable into two Hamilton double-rays. This seems to be an interesting question for which no results have been found by the author. Another potential generalisation comes from the interpretation of $G\square K_2$ as a prism. If instead we examine $G\square P_\infty$, where P_∞ is the double-ray $Cay(\mathbb{Z}, \pm\{1\})$, then $G\square P_\infty$ can be viewed as an infinite number of $G\square K_2$ prisms stacked on top of one another in both directions. Note, however, that as P_∞ is 2-regular, G must now be even-regular too for $G\square P_\infty$ to be even-regular. The question, then, is what conditions are required of G for $G\square P_\infty$ to be Hamilton decomposable? This will be investigated in the following section, but for completeness, we will briefly mention other reasonable generalisations. Note that for any other path P_n of length n, $G\square P_n$ will not be of regular degree, so it is not a suitable candidate for a Hamilton decomposition. Further note that if one lets G be infinite and Hamilton decomposable, then $G\square P_\infty$ is already covered by the result of [15] for the Cartesian product of two infinite graphs.

1.5 Preliminaries for $G\square P_\infty$

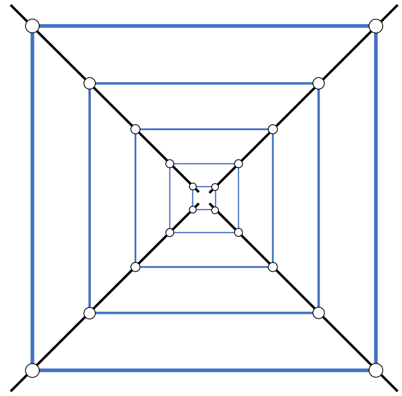
In keeping with the standard terminology used for finite Cartesian products involving K_2, we will refer to $G\square P_\infty$ as an *infinite prism*, with *base graph* G, the horizontal segments of a vertically oriented infinite prism. See the different perspectives of $C_4\square P_\infty$ provided in Figure 1.5. We present here a formal proof of the number of ends of the Cartesian product of a finite graph G with P_∞, as we use the fact that it is two-ended extensively throughout the thesis.

Claim 2. *$G\square P_\infty$ has two ends if G is finite and connected.*

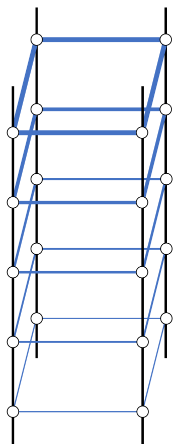
Proof. Let $E = \{\{(x, 0), (x, 1) \mid x \in G\}$ be the set of edges equivalent to the $\{0, 1\}$ edge in P_∞ which connect two copies of G in $G\square P_\infty$. E is an edge-cut set. Let $B = \{[(x, y), \dots] \mid y \leq 0, x \in G\}$ be the set of rays which start below E, and let $A = \{[(x, y), \dots] \mid y \geq 1, x \in G\}$ be the set of rays which start above E. Clearly, A and B partition the set of rays in $G\square P_\infty$. For a ray r, let $E_r = \{e \mid e \in E(r), e \in E\}$ be the set of edges of r in E. For a ray $a \in A$, let $a \in U$ if and only if $|E_a| \equiv 0 \pmod 2$, and for a ray $b \in B$, let $b \in U$ if and only if $|E_b| \equiv 1 \pmod 2$. If a or b is not a member of U, let it be a member of D. Note that $G\square P_\infty \setminus E$ has two connected components above and below the edge-cut E, and as E is finite, the number of edges of E in each ray is finite. If a ray starts above E and uses an even number of edges of E, then it must extend infinitely into the upper connected component of $G\square P_\infty \setminus E$, so we assign it to U. Likewise if a ray starts below E and uses an odd number of edges of E, it must also extend infinitely into the upper connected component of $G\square P_\infty \setminus E$, so we also assign it to U. Otherwise, the ray must extend into the lower connected component of $G\square P_\infty \setminus E$, so we assign it to D. Let $S = \{(x, y) \mid y \in \{0, 1\}, x \in G\}$ be a vertex-set, and note that U and D also partition the rays of $G\square P_\infty$ based on the connected component of $G\square P_\infty \setminus S$ into which the ray



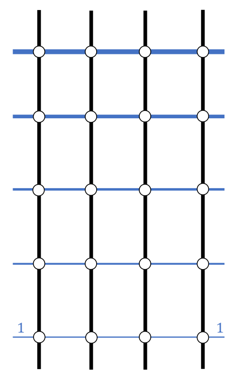
(a) Sideways. The individual cube sections are perhaps most clear here.



(b) Looking through the prism as if we are standing in the hallway of Hilbert's hotel. One end extends infinitely into the page, the other end extends outwards around us.



(c) Upright. The perspective in Figure 1.5d is easiest seen by unfolding and flattening this perspective.



(d) If the vertices of $C_4 \Box P_\infty$ are plotted on a Cartesian plane, we get this perspective. Note that, for example, the two edges labelled 1 are actually the same edge.

Figure 1.5: $C_4 \Box P_\infty$ from four different perspectives. In each, the blue lines are the C_4 edges, and the black lines are the P_∞ edges.

extends infinitely. S separates infinitely many vertices of any pair of rays from differing partitions, so the rays of U and D belong to different equivalence classes, so the number of ends of $G \Box P_\infty$ is at least two.

Now, without loss of generality show that every pair of rays s_1 and s_2 in U belong to the same equivalence class. Note that for every ray in U, there is an edge of the form $\{(x, y), (x, y+1)\}$ for some $x \in V(G)$ and every $y \in \mathbb{Z}^+$. If there were a $y' \in \mathbb{Z}^+$ such that such an edge did not exist, then the ray would not be infinite, as there are only a finite number of vertices between the y' and 0 levels, a contradiction as every ray is infinite. For the ray s_1, let $L_0 = \{\{(x, y), (x, y+1)\} \mid y \equiv 0 \pmod 2, x \in V(G)\} \subseteq E(s_1)$ be a subset of the edges of s_1 such that y is even, and let there be exactly one edge for each y. Note that L_0 is not unique as s_1 may use multiple such edges for a given y. If there is such a y, arbitrarily pick one of the edges. For the ray s_2, let $L_1 = \{\{(x, y), (x, y+1)\} \mid y \equiv 1 \pmod 2, x \in V(G)\} \subseteq E(s_2)$ be a subset of the edges of s_2 such that y is odd, and let there be exactly one edge for each y. Again note that L_2 is not unique. Together, $L_0 \cup L_1$ is a set of edges which can be ordered by their y-value, and alternate between being an edge of s_1

and s_2. Let s_3 be a ray which visits each edge of $L_0 \cup L_1$ in order. As G is connected, for each consecutive pair of edges $\{(x, y), (x, y+1)\} \in L_0$ and $\{(x', y+1), (x', y+2)\} \in L_1$, there is always a path from $(x, y+1)$ to $(x', y+1)$. Further, as s_3 intersects s_1 and s_2 in an infinite number of vertices, there is no finite set of vertices which separates them into distinct connected components of $G\Box P_\infty \setminus S$. Thus, s_1, s_2, and s_3 are all equivalent, and as s_1 and s_2 were arbitrary rays in U, every pair of rays in U are equivalent. The same argument can be applied to rays in D, and thus $G\Box P_\infty$ has exactly two ends.

□

1.5.1 Admissibility criteria for $G\Box P_\infty$

As mentioned already, for $G\Box P_\infty$ to be even-regular and connected, G must be even-regular and connected. The following criteria are a corollary of the proof of the admissibility criteria in Lemma 1, as $G\Box P_\infty$ is also two-ended.

Corollary 2. *For a $2(k-1)$-regular graph G, if $G\Box P_\infty$ is Hamilton decomposable, then G is connected and* $|G| \equiv k \pmod 2$

Proof. There are precisely $|G|$ edges in an edge-cut separating any two copies of G, so $|G| \equiv k \pmod 2$. As P_∞ is connected, $G\Box P_\infty$ is connected iff G is connected. □

1.6 The Cartesian connection

The following result provides a connection between Cayley graphs $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ with only one non-torsion element, and Cartesian products $G\Box P_\infty$ where G is a Cayley graph. This connection will be used throughout to derive results for both families of graphs.

Claim 3. *If $S^+ = \{g_1, \dots, g_k\}$, S generates $\Gamma = \Gamma_{\mathit{fin}} \oplus \mathbb{Z}$, $|g_i| = b_i$ for $1 \le i < k$, and $|g_k|$ is infinite, then $Cay(\Gamma, S)$ is isomorphic to the Cartesian product $G\Box P_\infty$ where $G = Cay(\Gamma_{\mathit{fin}}, \pm\{g'_1, \dots, g'_{k-1}\})$ and $g_i = (g'_i, 0)$ for all $1 \le i < k$.*

Proof. Let $g_k = (h^*, j)$ be the infinite-order generating element. Observe that an arbitrary element $x \in \Gamma$ is equal to some finite combination of the generating elements, so $x = \sum_{i=1}^{k} a_i g_i$ for $0 \le a_i < b_i$, $a_k \in \mathbb{Z}$. Then $x = h' + a_k g_k$ for some $h' \in \Gamma_{\text{fin}}$, so every vertex in $Cay(\Gamma, S)$ can be represented by $x = (h', 0) + a_k g_k = (h', 0) + a_k(h^*, j) = (h' + a_k h^*, a_k j) = (h, a_k j)$, an element of the Cartesian product $V(G) \times V(P_\infty)$, where $V(G) = \Gamma_{\text{fin}}$. Also note that S^+ only generates Γ if $j = 1$, so x can actually be represented as (h, ℓ) for $\ell = a_k \in \mathbb{Z}$. This also means that conversely, any element of $V(G) \times V(P_\infty)$ corresponds to a unique vertex in $Cay(\Gamma, S)$.

We now show that adjacency is preserved in the isomorphism, so two vertices are adjacent in $Cay(\Gamma, S)$ iff they are adjacent in $G\Box P_\infty$. Take a pair of vertices $x = (h, \ell), y = (h', j) \in Cay(\Gamma, S)$.

$$
\begin{aligned}
x \sim y \text{ in } Cay(\Gamma, S) &\iff x = y \pm g_i \text{ for some } g_i \in S^+ \\
&\iff (h, \ell) = (h', j) \pm g_i \\
&\iff (h, \ell) = (h' \pm h^*, j \pm 1) \text{ OR } (h, \ell) = (h' \pm g'_i, j) \text{ if } 1 \le i < k \\
&\iff h = h' \pm h^* \text{ and } \ell = j \pm 1 \text{ OR } \ell = j \text{ and } h = h' \pm g'_i \\
&\iff h = h' \pm h^* \text{ and } \ell \sim j \text{ in } P_\infty \text{ OR } \ell = j \text{ and } h \sim h' \text{ in } G \\
&\iff (h, \ell) \sim (h' \pm h^*, j) \text{ in } G\Box P_\infty,
\end{aligned}
$$

where $G = Cay(\Gamma_{\text{fin}}, \pm\{g'_1, \dots, g'_{k-1}\})$. Note that if $h^* \neq 0$, the two ends of the spoke edges between copies of G are not the same Γ_{fin} coordinate. However, due to the vertex-transitivity of every finite Cayley graph, this is still an isomorphism. □

This connection, and the underlying two-ended structure, suggests that similar techniques can be used for finding Hamilton decompositions of $G\Box P_\infty$ and $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ graphs, even when G is not a Cayley graph, or when $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ has more than one non-torsion generating element. For this reason, it will be convenient to explain techniques using the more general representation of an arbitrary two-ended, locally finite graph G_∞, where G_∞ could be the Cayley graph $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ or the Cartesian product $G \square P_\infty$ for a finite graph G. Note that the Cartesian notation for vertices of (x, y) is appropriate for both families of graphs, where $x \in \Gamma_{\text{fin}}$ and $y \in \mathbb{Z}$ for $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$, and $x \in V(G)$ and $y \in V(P_\infty) = \mathbb{Z}$ for $G \square P_\infty$. For $Cay(\Gamma, S)$, let $w = |\Gamma_{\text{fin}}|$ be the size of the largest finite subgroup, and for $G \square P_\infty$, let $w = |V(G)|$ be the size of the base graph.

1.7 Notation and Techniques

1.7.1 Techniques and thesis-specific notation

Note that both $G \square P_\infty$ and $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ graphs have a vertical translation symmetry, where $f_i : V(G_\infty) \mapsto V(G_\infty)$, $f_i((x, y)) = (x, y + i)$ for all $i \in \mathbb{Z}$ is an automorphism of the graph. It therefore makes sense to exploit this symmetry through the use of periodic construction techniques. By-and-large, the techniques used for constructions in this thesis will involve finding a finite path, called the *canonical* path, which can be translated repeatedly to form a *periodic* double-ray, called the *canonical* double-ray. If this canonical double-ray can be translated again to give the other double-rays in the Hamilton decomposition, the canonical double-ray will be called a *periodic translation* solution. Here we define the non-standard notation used in the thesis, and introduce the techniques and ideas used to find solutions.

Let G_∞ be a graph which is either a Cayley graph of the form $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ or a Cartesian product graph of the form $G \square P_\infty$. For a path $P = [(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)]$ in G_∞, we define $P + i$ as $[(x_1, y_1 + i), (x_2, y_2 + i), \ldots, (x_n, y_n + i)]$, the path P translated vertically by $(0, i)$. Note that $P + i$ is the image of P under the action f_i. For a finite sequence of internally vertex-disjoint paths $P_1, P_2, \ldots P_n$, we write the concatenation of these paths as $[P_1, P_2, \ldots, P_n]$. This is a path, as long as the end vertex of P_i is equal to the start vertex of P_{i+1} for all $1 \leq i < n$. For a path P which starts at (x_1, y_1) and ends at (x_2, y_2), we define the *vertical displacement* of P as $y_2 - y_1$. Additionally, for an infinite set of internally vertex-disjoint paths $\{P + di, \forall i \in \mathbb{Z}\}$, with d fixed, we write their infinite concatenation as $\bigcup_{i \in \mathbb{Z}} P + di$, and note that if the end vertex of $P + di$ is equal to the start vertex of $P + d(i + 1)$, then this infinite union is a double-ray.

Definition 5 (Period of a double-ray)**.** A double-ray Q is periodic with period $d > 0$ if $Q + d = Q$, and d is minimal.

1.7.2 Sections and finite reductions

In order to ensure that a canonical path can be translated to create a canonical double-ray, and translated again to partition the edge-set of G_∞, we define the following notions of a finite segment of G_∞, as well as some useful terminology. The first primarily exists as an easier segment on which to build canonical paths, as long as the underlying infinite graph admits a canonical path on this easier type of segment.

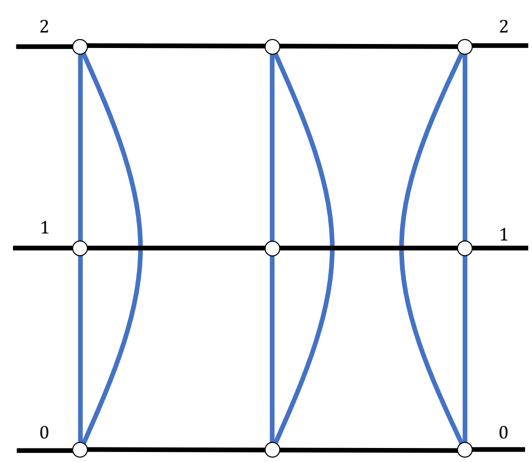


Figure 1.6: A 3-finite reduction of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0, 1), (1, 0)\})$.

Definition 6 ((d, a)-section)**.** A (d, a)-*section* of $G \square P_\infty$ is defined to be the subgraph of $G \square P_\infty$ induced by a set of vertices $\{(x, y) \mid a \leq y < a + d, x \in V(G)\}$.

Where it is clear, *the d-section* is the $(d, 1)$-section. Any $(1, a)$-section is isomorphic to a base graph G. We will also refer to each $(1, a)$-section as level a, or a copy of G. We define a *spoke*, or more specifically an *m-spoke*, to be the subgraph of $G \square P_\infty$ induced by a set of vertices $\{(m, y), \forall y \in \mathbb{Z}\}$, for a fixed $m \in V(G)$. Each m-spoke is isomorphic to P_∞.

Definition 7 (Finite reduction of $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$)**.** The *$d$-finite reduction* of a two-ended infinite Cayley graph $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ is the finite graph with vertex set $V = \{(x, y) \mid x \in \Gamma_{\text{fin}}, y \in \mathbb{Z}_d\}$, and edge set $E = \{\{(x, y), (x + g_x, y + g_y \mod d)\} \mid (x, y) \in V, (g_x, g_y) = g \in S\}$. See Figure 1.6 for an example.

For an edge $e = \{(x, y), (x + g_x, y + g_y \mod d)\}$ in the d-finite reduction of $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$, let E_e be the set of edges of $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ of the form $\{(x, y'), (x + g_x, y' + g_y)\}$ for all $y' \equiv y \mod d$. We say that an edge e in the d-finite reduction is equivalent to every edge $e' \in E_e$.

Definition 8 (Finite reduction of $G \square P_\infty$)**.** The *d-finite reduction* of a two-ended infinite Cartesian product $G \square P_\infty$ is the finite graph $G \square C_d$, where $V(C_d) = \mathbb{Z}_d$.

For an edge $e = \{(x, y), (u, v)\}$ in the d-finite reduction of $G \square P_\infty$, it is convenient to break up the definition of E_e into cases. Note that for two vertices (x, y) and (u, v) to be adjacent in $G \square C_d$, either $x = u$ and $y \sim v$ in C_d, or $y = v$ and $x \sim u$ in G. If $e = \{(x, y), (u, y)\}$, let E_e be the set of edges of $G \square P_\infty$ of the form $\{(x, y'), (u, y')\}$ for all $y' \equiv y \pmod d$. If $e = \{(x, y), (x, y + 1)\}$ for $y \in \{0, 1, \ldots, d - 2\}$, let E_e be the set of edges of $G \square P_\infty$ of the form $\{(x, y'), (x, y' + 1)\}$ for all $y' \equiv y \pmod d$. If $e = \{(x, d - 1), (x, 0)\}$ then E_e is the set of edges of $G \square P_\infty$ of the form $\{(x, y'), (x, y' + 1)\}$ for all $y' \equiv -1 \pmod d$.

We say that an edge e in the d-finite reduction is equivalent to every edge $e' \in E_e$. For a path in a d-finite reduction of G_∞, we say that P is equivalent to a path P' in G_∞ where P' is a path of the same length as P, with edges which are equivalent to the corresponding edges in the path P. It will also be convenient to have a notion of equivalence classes for edges, that makes use of the symmetry of G_∞.

Definition 9 (Vertical translation of an edge)**.** For an edge $e = \{(x_1, y_1), (x_2, y_2)\}$ in either $G \square P_\infty$ or $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ and an integer i, $e + i = \{(x_1, y_1 + i), (x_2, y_2 + i)\}$ is the vertical translation of e by i. Note that $e + i$ is also the image of e under the action f_i.

Definition 10 (Edge-types)**.** We say two edges e_1 and e_2 in either $G \square P_\infty$ or $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ are of the same edge-type if $e_1 = e_2 + i$ for some integer i. Note that, in a multi-graph where e_1 and e_2 have the same end-points, we say e_1 and e_2 are not of the same edge-type. For two other edges e_3 and e_4 which have the same end-points and where $e_3 = e_1 + i = e_2 + i$, we arbitrarily assign e_3 as being of the same edge-type as e_1 and e_4 as being of the same edge-type as e_2. This preserves the notion of each edge-type appearing exactly d times in a d-finite reduction, and the uniqueness of the translation action $e + i$ on an edge e. (See Figure 1.7.)

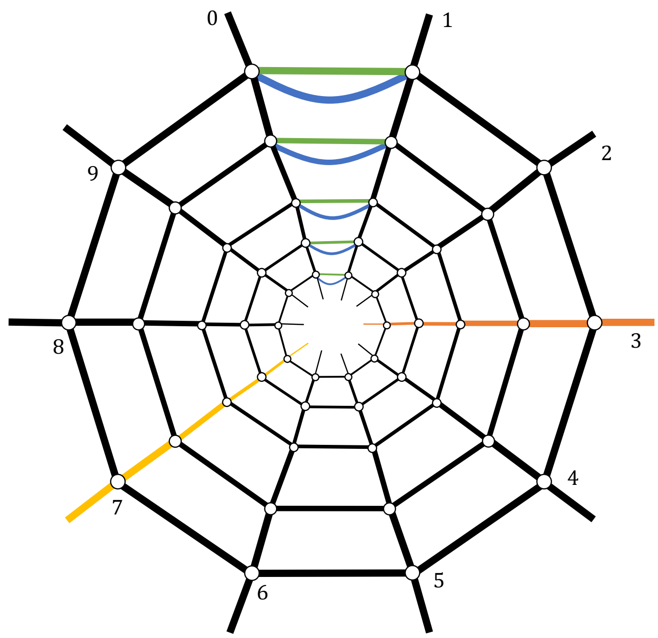
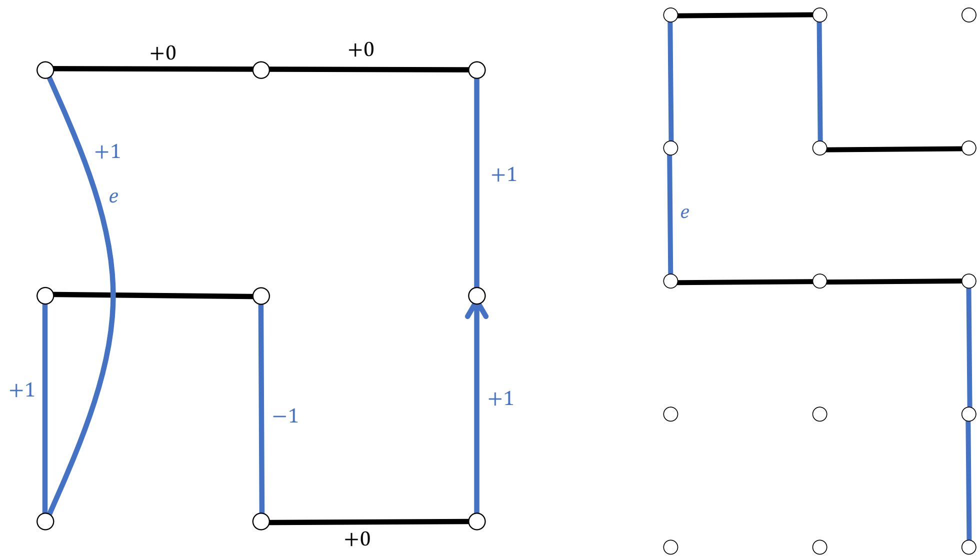


Figure 1.7: Each colour represents a set of edges of the same edge-type. Note that as only strictly vertical translations are allowed, the yellow and orange edges are not of the same type, and the green and blue multi-edge pairs are of different types.

Note that an edge in $G \square P_\infty$ is either equivalent to $\{(u, a), (v, a)\}$ for some edge uv, or $\{(m, a), (m, a+1)\}$ for some $a \in \mathbb{Z}$. An edge of the first form will be called a uv-edge, and an edge of the second form will be called an m-spoke edge. All the uv-edges for a fixed uv are of the same type, and all the m-spoke edges for a fixed m are of the same type.

Definition 11 (Net vertical displacement in G_∞)**.** Take an edge $e' = \{(x_1, y_1), (x_2, y_2)\}$ in a path P' in G_∞, and note that it is can be assigned a direction determined by the direction of the path. Suppose e corresponds to the directed edge $((x_1, y_1), (x_2, y_2))$. Its net vertical displacement is $\delta(e) = y_2 - y_1$. Note that the net vertical displacement is fixed in its absolute value, but the sign is dependent on the direction of e'. For a path P' in G_∞ with fixed direction and edge-set $E(P')$, let the net vertical displacement of P' be $\delta(P') = \sum_{e' \in E(P')} \delta(e)$. Again note that the net vertical displacement is fixed in its absolute value, but the sign is dependent of the direction of P'. If (x_1, y_1) is the start and (x_2, y_2) the end of P', then the net vertical displacement of P' is also $\delta(P') = y_2 - y_1$. We define the net vertical displacement of edges and paths in a d-finite reduction of G_∞ in a way that keeps track of the net vertical displacement of the edges and paths to which they are equivalent in G_∞.



(a) An example of a cycle Q in the 3-finite reduction of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,0)\})$.

(b) The path Q' in $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,0)\})$ to which Q is equivalent.

Figure 1.8

Definition 12 (Net vertical displacement in a d-finite reduction)**.** For an edge e in a path P in a d-finite reduction of G_∞, let e' be an edge in E_e, and let P' be a path equivalent to P such that $e' \in E(P')$. The net vertical displacement of e is $\delta(e) = \delta(e')$, and the net vertical displacement of P is $\delta(P) = \delta(P') = \sum_{e \in P} \delta(e)$. As in G_∞, the net vertical displacement is fixed in its absolute value, but the sign depends on the direction.

Note that the net vertical displacement of a cycle in a finite reduction is not necessarily 0. For the example in Figure 1.8a, start at the bottom-right vertex and follow the path counter-clockwise as indicated by the arrow. The first two blue edges both have net vertical displacement of $+1$. Then the black edges have net vertical displacement of 0. Then the edge labelled e' has a net vertical displacement of $+1$, not the -2 one might expect visually, as e' is equivalent to the $(0,1)$-edge e in $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,0)\})$. (See Figure 1.8b for the path Q' to which Q is equivalent.) Then the final two blue edges have net vertical displacement of $+1$ and -1 respectively, making the net vertical displacement for the cycle and the path it is equivalent to $+3$.

1.8 Sufficient criteria for a solution

The following claim provides a set of sufficient criteria for finding a periodic translation solution for G_∞.

Claim 4. *A two-ended, 2k-regular infinite graph G_∞ has a Hamilton decomposition into k periodic double-rays, where each double-ray is equivalent under vertical translation, if a single Hamilton cycle H on a kh-finite reduction exists such that:*

1. *Each edge type is included exactly h times in H.*
2. *For each edge $e \in E(H)$, $e+h, e+2h, \ldots, e+(k-1)h$ are not in $E(H)$.*
3. *The net vertical displacement of H is kh or $-kh$.*

Proof. Assume we have a Hamilton cycle H which satisfies all the conditions. First, show that H is equivalent to a double-ray Q_0. For each edge $e' \in E(H)$ and $i \in \mathbb{Z}$, let $e + ikh$ be in $E(Q_0)$, where e is the edge in G_∞ equivalent to e'. Note that for a fixed i, the set of edges $e+ikh$ for all $e' \in E(H)$ forms a path from a vertex (x, y) to $(x, y + kh)$, as the net vertical displacement of H is kh or $-kh$ due to (3). As H is Hamiltonian, every vertex in G_∞ has valency 2. H connected and (3) implies that for any $v \in V(G_\infty)$, there is a path from v to $v + kh$ and $v - kh$, and thus from any vertex to any other vertex. Thus, Q_0 is connected, and so Q_0 is a Hamilton double-ray.

Now show that Q_0 can be translated by $h, 2h, \ldots, (k-1)h$ to generate the other double-rays in the decomposition $Q_1, Q_2, \ldots, Q_{k-1}$. Equivalently, show that $E(Q_0) \cup E(Q_1) \cup \cdots \cup E(Q_{k-1}) = E(G_\infty)$ and $E(Q_i) \cap E(Q_j) = \emptyset$ for $i \neq j$. First, assume for contradiction that $e \in E(Q_i) \cap E(Q_j)$ for some $i \neq j$. Without loss of generality, assume that that $e \in E(Q_0)$ and $e \in E(Q_i)$. Then $e = e' + ih$ for some $e' \in E(Q_0)$. But by (2), for each edge e' in H, and thus in Q_0, $e' + ih$ is not in Q_0. Contradiction. Thus, the pairwise intersections of the paths are empty. Then, show that for any $e \in E(G_\infty)$, $e \in E(Q_0) \cup E(Q_1) \cup \cdots \cup E(Q_{k-1})$. Consider the equivalence class of e in the kh-finite reduction. By (1), for each edge type, there are h edges in H. Consider the equivalent kh-finite reductions of $Q_1, \ldots, Q_{k-1}$, call them $H_1, \ldots, H_{k-1}$ respectively. It has already been shown that there is no intersection between the edge sets of the paths, so for each edge type, there are kh edges of each type in $E(H) \cup \cdots \cup E(H_{k-1})$, and kh edges in the kh-finite reduction. Thus, every edge type is in the union of the paths. Thus, $Q_0, \ldots, Q_{k-1}$ form a Hamilton decomposition into double-rays. ☐

A path P in an hk-finite reduction (not necessarily a cycle with net vertical displacement kh or $-kh$) will be called a partial or candidate solution, and if this path is also a cycle which satisfies Claim 4, then P will be called a solution. When representing partial solutions in a d-finite reduction, more than d layers may be drawn in order to better represent the net vertical displacement of the path.

1.9 Necessity of sufficient criteria

We now investigate the necessity of the criteria in Claim 4, under the assumption that the decomposition is periodic and each double-ray is equivalent under translation. Each edge type being used exactly h times is easily shown as otherwise certain edges will be over or under-represented in the partition as the paths are equivalent under translation. The net vertical displacement being exactly kh or $-kh$ also comes easily from the proof of Claim 4, as otherwise translating by hk will not lead to a double-ray.

Period is hk for some h

Claim 4 asserts that the solution is found in a kh-finite reduction, for some integer h. The following lemma proves that the period of the double-rays is a multiple of k, and thus it is necessary only to search for solutions in kh-finite reductions.

Lemma 2. *If a Hamilton decomposition into k periodic double-rays exists such that each double-ray is a translation of a double-ray Q_0, then a period of Q_0 can be found in an hk-finite reduction for some $h \in \mathbb{Z}^+$.*

Proof. As each double-ray $Q_1, \dots, Q_{k-1}$ is a translation of Q_0 by $t_1, \dots, t_{k-1}$, they must all have the same period, let b be the period of Q_0. Then without loss of generality we can say that $0 < t_j < b$ for all t_j. Assume that, $b > k$, as if $b = k$ then a period of Q_0 can be found in a k-finite reduction, with $t_j = j$, and if $b < k$ then between equivalent edges of Q_0 there are not enough edges for the other double-rays. Consider the edges e and $e+b$, both of which belong to Q_0. Then consider the $b-1$ edges in $M = \{e+1, \dots, e+b-1\}$. If none of these edges are in Q_0, then as each other double-ray is a translation of Q_0 by some integer less than b, only one edge of each of the $k-1$ remaining double-rays is in M. But $k-1 < b-1$ so not a Hamilton decomposition. Thus $e+i$ is in Q_0 for some $e+i \in M$, and each Q_j $(0 < j \leq k)$ also has exactly one additional edge in M, either $e+i+t_j$ if $i+t_j < b$ or $e+i-b+t_j$ if $i+t_j > b$. Note that $i+t_j \neq b$ as $e+b$ is already in Q_0. So the b edges in $M \cup \{e\}$ are divisible by the number of double-rays k, so $b = hk$ for some positive integer h. □

Translation by h and $2h$ in the 6-regular case

The following claim tells us that for a given $3h$-finite reduction, the translations of any solution are by h and $2h$, or the solution is a repeated solution on a smaller finite reduction. Thus, this criterion is also necessary, provided that any smaller $3h$-finite reductions have already been searched, and a solution either found or ruled out.[1]

Lemma 3. *Suppose that Q_0, Q_1, Q_2 is a Hamilton decomposition of G_∞ and that*

- *Q_0 is a Hamilton double-ray of period $3h$;*
- *$Q_1 = Q_0 + t_1$ and $Q_2 = Q_1 + t_2$.*

Then $t_1 = t_2 = \pm h$.

Proof. Let Q_0, Q_1, Q_2 be three Hamilton double-rays where $Q_1 = Q_0 + t_1$, $Q_2 = Q_1 + t_2$ and let t_3 be such that $Q_0 = Q_2 + t_3$. Since the double-rays have period $3h$, we have $t_1 + t_2 + t_3 = 3h$. Let e be an edge of Q_0 and let $E = \{e, e+1, e+2, \dots, e+3h-1\}$ be the edges in the equivalence class of e in a $3h$-section. Since $e \in E(Q_0)$, we have $e + t_1 \in E(Q_1)$ and $e + t_1 + t_2 \in E(Q_2)$.

Now consider the edge $e - t_2$. If $e - t_2 \in E(Q_1)$ then $e \in E(Q_2)$ which contradicts the fact that $e \in E(Q_0)$. If $e - t_2 \in E(Q_0)$ then $e - t_2 + t_1 \in E(Q_1)$ and hence $e + t_1 \in E(Q_2)$ which contradicts the fact that $e + t_1 \in E(Q_1)$. Hence $e - t_2 \in E(Q_2)$. Thus we have $e - 2t_2 \in E(Q_1)$. Therefore, for each $e \in E(Q_0)$ we have $e - 2t_2 - t_1 \in E(Q_0)$, or equivalently, since $t_1 + t_2 + t_3 = 3h$, we have $e - t_2 + t_3 \in E(Q_0)$. This implies that either $t_2 = t_3$ or the period of Q_0 is a proper factor of $3h$. This latter is a contradiction, so we have that $t_2 = t_3$.

Now consider the edge $e+t_2$. If $e+t_2 \in E(Q_2)$ then $e \in E(Q_1)$ which contradicts the fact that $e \in E(Q_0)$. If $e + t_2 \in E(Q_0)$ then $e + t_1 + t_2 \in E(Q_1)$ which contradicts the fact that $e + t_1 + t_2 \in E(Q_2)$. Hence $e + t_2 \in E(Q_1)$. Therefore, for each $e \in E(Q_0)$, we have $e + t_2 - t_1 \in E(Q_0)$. This implies that either $t_2 = t_1$ or the period of Q_0 is a proper factor of $3h$. This latter is a contradiction, so we have that $t_1 = t_2 = t_3$. Thus since $t_1 + t_2 + t_3 = 3h$ we have $t_1 = t_2 = \pm h$. □

These results show us that, if a periodic translation solution of any length exists, it is guaranteed to be found using the criteria established above. If a well-designed search algorithm fails to find a solution, it can then be stated that no such translation solution exists for a given kh-finite reduction or period. This procedure says nothing about the existence of non-translation solutions, for which there is reason to suspect that a solution always exists. (See Section 4.4 for a particular example).

[1] This proof may be generalisable to any k, but given that the thesis has focused on 6-regular graphs, this generalisation has not been actively pursued.

Chapter 2

$G \Box P_\infty$ graphs

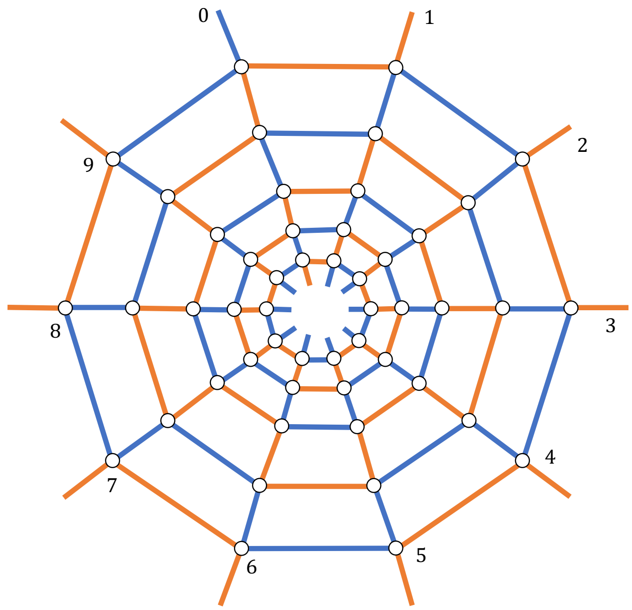


Figure 2.1: $C_{10} \Box P_\infty$ decomposed into two Hamilton double-rays. The x-value is represented around the outside, and the inner-most level is level 0.

If G is a finite Hamilton decomposable graph, then the infinite prism $G \Box P_\infty$ represents an easy unsolved case of the generalisation of Bermond's Conjecture. If G is a finite Cayley graph, then it is conjectured to have a Hamilton decomposition, and a Hamilton decomposition of $G \Box P_\infty$ would also derive results for two-ended Cayley graphs due to Claim 3. In this chapter, we derive various results on the Hamilton decomposition of $G \Box P_\infty$ when G is Hamilton decomposable, and determine their consequences for two-ended Cayley graphs. As all but one of the edge-types in a $G \Box P_\infty$ graph is an edge which remains inside one copy of G, we will represent the graphs in a circular fashion, as in Figure 2.1. Each copy of G will be represented with vertices in concentric circles, ordered according to one of the Hamilton cycles, with spoke edges joining equivalent vertices in each layer. Though the example base graphs are Cayley graphs, the given results apply for any graph with a Hamilton decomposition, not just Cayley graphs.

2.1 2-valent base graphs

The only connected, 2-valent, finite graphs are where $G \simeq C_n$. Let $C_n = (0, 1, 2, 3, \ldots, n-2, n-1)$. By Corollary 2, if $C_n \Box P_\infty$ is to be Hamilton-decomposable into two double-rays, then n must be even. Q is constructed like so:

$$Q = [(0,1), (1,1), (1,2), (2,2), (2,1), (3,1), \ldots, (n-2,1), (n-1,1), (n-1,2), (0,2), (0,3)]$$

Note that the only spoke edge not used in the $(2,1)$-section is on the 0-spoke, so the number of spoke edges used is $n-1$ in the $(2,1)$-section, and one in the $(2,2)$-section. Let $Q_0 = \bigcup_{i \in \mathbb{Z}} Q + 2i$, and $Q_1 = Q_0 + 1$.

$$\begin{aligned} E(Q_0) = &\{\{(x,y),(x+1,y)\} \mid x \equiv y+1 \pmod 2\} \\ &\cup \{\{(x,y),(x,y+1)\} \mid y \equiv 1 \pmod 2, x \neq 0\} \\ &\cup \{\{(0,y),(0,y+1)\} \mid y \equiv 0 \pmod 2\}, \end{aligned}$$

and

$$
\begin{aligned}
E(Q_1) = & \{\{(x,y),(x+1,y)\} \mid x \equiv y \pmod 2\} \\
& \cup \{\{(x,y),(x,y+1)\} \mid y \equiv 0 \pmod 2, x \neq 0\} \\
& \cup \{\{(0,y),(0,y+1)\} \mid y \equiv 1 \pmod 2\}.
\end{aligned}
$$

Therefore, $E(Q_0) \cap E(Q_1) = \emptyset$, and $E(Q_0) \cup E(Q_1) = E(C_n \square P_\infty)$. Clearly, Q_0 is also connected, as the first vertex of $Q+2$ is the vertex $(0,3)$, the last vertex of Q, so Q_0 and Q_1 form a Hamilton decomposition into double-rays. An example construction on $C_{10} \square P_\infty$ is given in Figure 2.1.

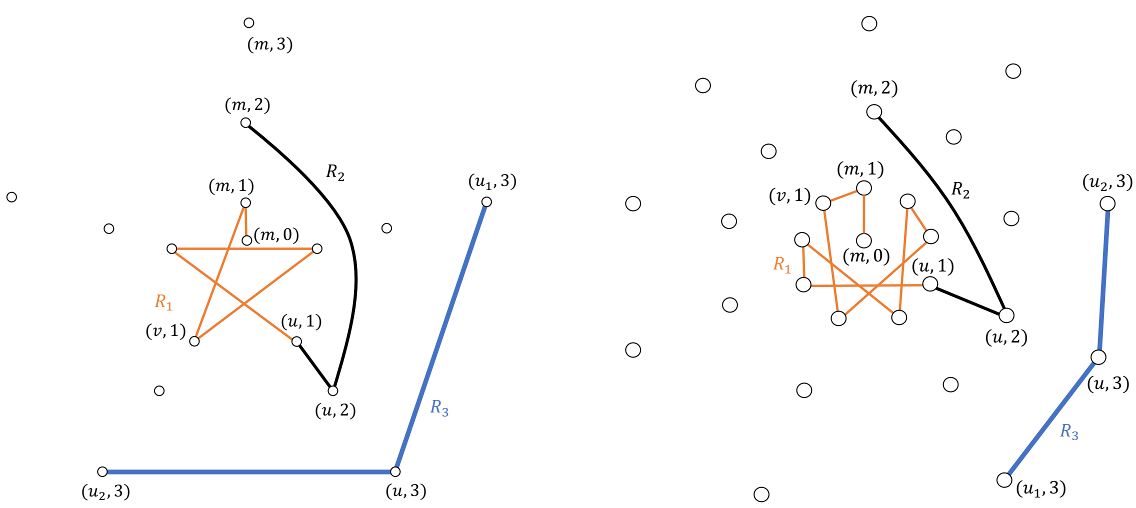
2.2 4-valent base graphs

For a 4-valent base graph G, $G \square P_\infty$ will have valency 6, so by Corollary 2, for $G \square P_\infty$ to be Hamilton-decomposable into double-rays, $|V(G)|$ must be odd.

Theorem 5. *If G is a 4-valent graph of odd order with a Hamilton decomposition, then $G \square P_\infty$ is Hamilton decomposable.*

Proof. Suppose G is an odd order 4-regular graph with a Hamilton decomposition into two cycles H_1 and H_2. The proof is by construction. Consider the 3-section, with vertices the elements of the set $\{(x,y) \mid x \in V(G), 1 \leq y < 4\}$. The aim is to construct, from smaller paths, a canonical path Q between $(m,0)$ and $(m,3)$ which contains each type of edge once, and visits each vertex exactly once, for some arbitrary $m \in V(G)$. Let u and v be the vertices adjacent to m in H_1.

Let $R_1 = [(m,0),(m,1),(v,1),\ldots,(x,1),(x',1),\ldots,(u,1)]$ for all xx'-edges in H_1, except the um-edge. Let $R_2 = [(u,1),(u,2),(m,2)]$. R_1 and R_2 can be viewed as the edges of H_1 embedded in level 1 with the edge $\{(u,1),(m,1)\}$ moved into level 2, and the initial m-spoke edge from $(m,0)$ to $(m,1)$ appended. Note that $[R_1, R_2]$ is a path, and visits each vertex in level 1 exactly once. Each edge-type from H_1 is used exactly once, and a single u-spoke edge and a single m-spoke edge are also used exactly once. An example for $K_5 \square P_\infty$ is given in Figure 2.2a, and for $Cay(\mathbb{Z}_9, \pm\{1,3\}) \square P_\infty$ in Figure 2.2b.



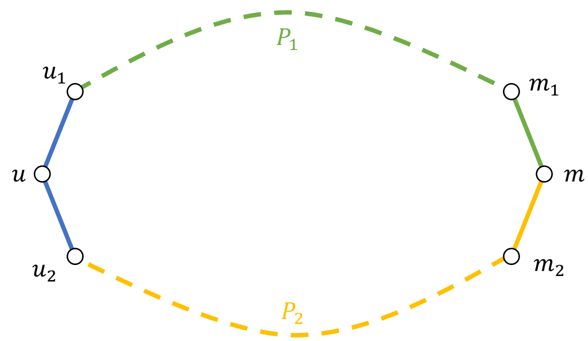
(a) $K_5 \square P_\infty$.

(b) $Cay(\mathbb{Z}_9, \pm\{1,3\}) \square P_\infty$.

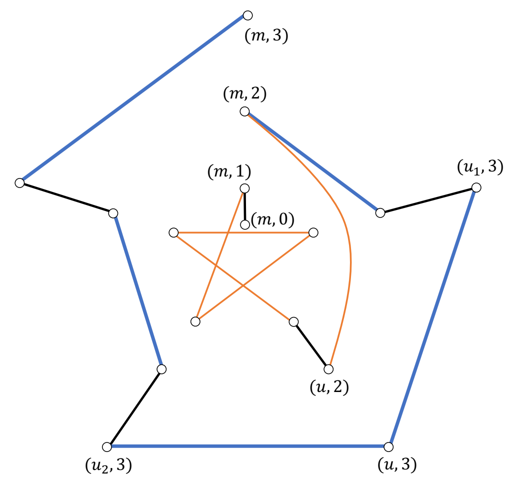
Figure 2.2: The paths R_1 (orange, thinner), R_2 (black), R_3 (blue, thicker).

Now consider the edges of H_2, and denote by u_1 and u_2 the vertices adjacent to u in H_2. Let $R_3 = [(u_1,3),(u,3),(u_2,3)]$. R_3 includes the edges u_1u and u_2u in level 3 to avoid the already visited vertex $(u,2)$. (See Figure 2.2)

Note in order to use exactly one of each edge-type, there are $n-2$ edge-types from H_2 and $n-2$ spoke edges which are not yet present in Q. The remaining H_2 edges belong to two paths, $P_1 = [u_1, a_1, \ldots, m_1, m]$ and $P_2 = [u_2, a_2, \ldots, m_2, m]$ (see Figure 2.3a). As n is odd, exactly one of these two paths will be of even length, and the other odd. Let P_1 be the odd length path, and P_2 the even length path. From P_1, generate $P_1' = [(u_1, 3), (u_1, 2), (a_1, 2), (a_1, 3), \ldots, (m_1, 3), (m_1, 2), (m, 2)]$. P_1' alternates the edges from P_1 between level 2 and 3, joining them with spoke edges. As P_1 has an odd number of edges, P_1' has an odd number of spoke edges, so it starts in level 3 and ends in level 2, connecting to $(m, 2)$, the last vertex of R_2.



(a) The H_2 cycle (blue) with some vertices and paths labelled to aid the construction.



(b) The canonical path Q_0 for $K_5 \square P_\infty$, with colours representing the original cycles in K_5. H_1 is orange and thinner, H_2 is blue and thicker.

Figure 2.3: Supporting sketches for the 4-valent construction.

Similarly, use P_2 to construct $P_2' = [(u_2, 3), (u_2, 2), (a_2, 2), (a_2, 3), \ldots, (m_2, 2), (m_2, 3), (m, 3)]$. As P_2 has an even number of edges, P_2' has an even number of spokes, so it starts and ends in level 3. Let $Q = [R_1, R_2, \hat{P}_1', R_3, P_2'], Q_0 = \bigcup_{i \in \mathbb{Z}} Q + 3i, Q_1 = Q_0 + 1, Q_2 = Q_0 + 2$. By construction, each edge-type in H_1 and H_2 occurs in Q exactly once, and an edge from each spoke occurs in Q exactly once: in R_1 for the m-spoke, in R_2 for the u-spoke, and in exactly one of P_1' or P_2' for each other w-spoke, depending on whether the vertex w was in P_1 or P_2. R_1 visits each level 1 vertex exactly once. For each vertex a not equal to u or m, a appears in exactly one of P_1 or P_2, so has a spoke edge in P_1' or P_2' respectively, and visits both $(a, 2)$ and $(a, 3)$ exactly once. As for u and m, $(u, 2), (m, 2) \in V(R_2)$, $(u, 3) \in V(R_3)$ and $(m, 3) \in V(P_2')$. Thus Q_0 is spanning, connected, and 2-regular, and so $E(Q_i) \cap E(Q_j) = \emptyset$ for all $i \neq j$, and $E(Q_0) \cup E(Q_1) \cup E(Q_2) = E(G \square P_\infty)$. Thus, Q_0, Q_1, Q_2 forms a Hamilton decomposition of $G \square P_\infty$.

□

Note that Bermond, Favaron and Maheo [7] proved that every 4-regular Cayley graph of a finite abelian group is Hamilton decomposable, and it was shown with Claim 3 that the Cayley graph $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, \pm S^+)$ of every two-ended abelian group where S^+ has exactly one non-torsion generating element is isomorphic to the Cartesian product of a finite Cayley graph $Cay(\Gamma_{\text{fin}}, S')$ with the double-ray P_∞. Together with Theorem 5, the following Theorem comes out as a corollary:

Theorem 6. *If $Cay(\Gamma_{fin} \oplus \mathbb{Z}, \pm S^+)$ is a 6-regular two-ended Cayley graph and S^+ has exactly one non-torsion generating element, then $Cay(\Gamma_{fin} \oplus \mathbb{Z}, \pm S^+)$ it has a Hamilton decomposition if and only if $|\Gamma_{fin}| \equiv 1$* (mod 2)*.*

2.3 6-valent base graphs

A 6-valent base graph implies that $G \square P_\infty$ is 8-valent, so it will be decomposed into four Hamilton double-rays. Thus, by Corollary 2, $|V(G)|$ must be even.

Theorem 7. *If G is a 6-valent graph of even order with a Hamilton decomposition, then $G \square P_\infty$ is Hamilton decomposable.*

The proof of the theorem relies on the following claim:

Claim 5. *If H_1, H_2, H_3 forms a Hamilton decomposition of G, then a triple of vertices $u, v, m \in V(G)$ exists such that:*

1. *u and v are an even distance apart in H_2*
2. *$v \sim m$ in H_1*
3. *$u \sim m$ in H_3*
4. *$u \neq v$*

Proof. For (4), as $vm \in H_1$ and $um \in H_3$, $u = v$ would imply the same edge appears in H_1 and H_3, a contradiction.[1]

For (1, 2, 3), assume for contradiction that:

$$\forall m \text{ such that } v \sim m \text{ in } H_1 \text{ and } u \sim m \text{ in } H_3, v \text{ and } u \text{ are an odd distance apart in } H_2. \quad (\dagger)$$

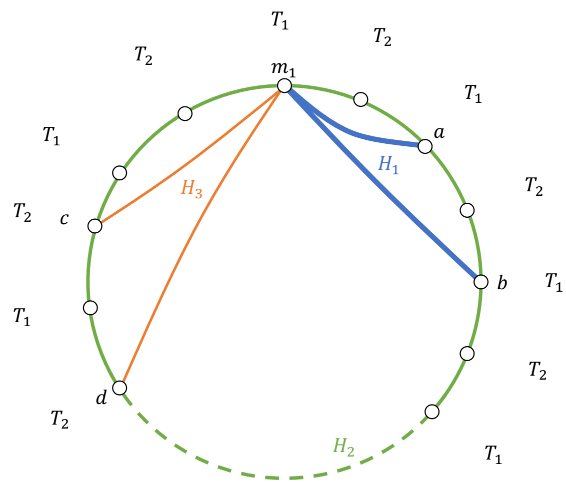


Figure 2.4: An illustration of the proof of the existence of the even distance assumption. H_1 is blue and thick, H_2 is the circle in green, and H_3 is orange and thin. The vertices are labelled as members of either T_1 or T_2 around the outside

As H_2 is an even length cycle, it is bipartite, so $V(G)$ can be partitioned into sets T_1 and T_2, such that any pair of vertices in the same set are an even distance apart in H_2. (See Figure 2.4)

For an arbitrary vertex $m_1 \in V(G)$, let m_1 be adjacent to a and b in H_1. Both a and b must be in the same set, either T_1 or T_2, otherwise any vertex c adjacent to m_1 in H_3 would be an even distance from either a or b in H_2, which would violate (†). Without loss of generality, let $a, b \in T_1$. Similarly, for m_1 adjacent to c and d in H_3, c and d must also be in the same set, either T_1 or T_2. If $a, b, c, d \in T_1$, then (†) would be violated, as a, c, m_1 would satisfy the conditions of the claim, so c, d must be in T_2. Without loss of generality, let m_1 be in T_1. The same argument forces all H_1 neighbours of a and b into T_1, and so on for all vertices in the same component as m_1 in H_1, so there is no path in H_1 from m_1 to a vertex in T_2, a contradiction as H_1 is spanning and connected.

□

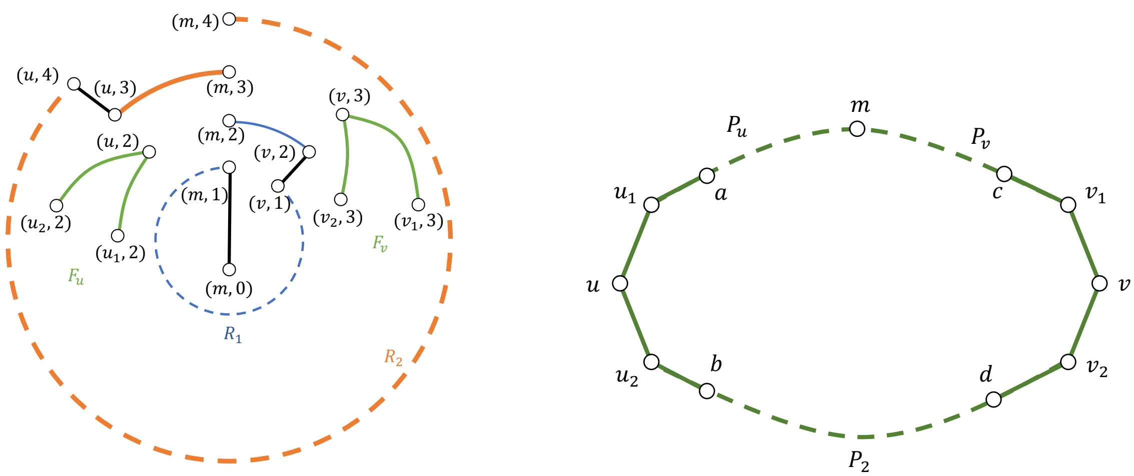
[1] Note that this precludes the possibility that G is a multi-graph decomposed into Hamilton cycles, however, with some alteration, the claim and its use should be recoverable to include multi-graphs. A triple u, v, m might still exist in a particular multi-graph, (they are typically not hard to find) but it is currently not guaranteed by the proof.

Proof of Theorem 7

Proof. Let the triple of vertices $u, v, m \in V(G)$ satisfy the conditions of Claim 5. Similarly to Section 2.2, the aim is to construct a canonical path Q between $(m, 0)$ and $(m, 4)$ that visits each vertex on the 4-section $\{(x, y) \mid x \in V(G), 1 \le y \le 4\}$, and contains exactly one of each type of edge. Let

$$R_1 = [(m, 0), (m, 1), \ldots, (x, 1), (x', 1), \ldots, (v, 1), (v, 2), (m, 2)]$$

for all xx'-edges in H_1 except the vm-edge. R_1 can be viewed as all H_1 edge-types embedded in level 1 with the vm-edge shifted to level 2 and connected to the level 1 edges with a v-spoke edge (See Figure 2.5a). R_1 is clearly a path and visits each vertex in level 1 exactly once.



(a) The partially constructed canonical path Q after the H_1 and H_3 edges are assigned to their levels. H_1 edges are inside in blue, H_2 green, H_3 outside and orange. The green edges are *forced* into their respective positions by the ends of R_1 and R_2.

(b) The important vertices of the base graph G, and the three paths which we use to construct Q.

Figure 2.5

Similarly, define R_2:

$$R_2 = [(m, 3), (u, 3), (u, 4), \ldots, (x, 4), (x', 4), \ldots, (m, 4)]$$

for all xx'-edges in H_3 except the um-edge. As with R_1, R_2 can be viewed as all H_3 edges embedded in level 4 with the um edge shifted to level 3 and connected by a u-spoke edge to the level 4 edges (See Figure 2.5a). R_2 is also clearly a path and visits each vertex in level 4 exactly once. Let u_1 and u_2 be adjacent to u in H_2, and v_1 and v_2 be adjacent to v in H_2. Let $F_u = [(u_1, 2), (u, 2), (u_2, 2)]$ and $F_v = [(v_1, 3), (v, 3), (v_2, 3)]$. The construction so far appears as in Figure 2.5a.

There are now $n - 4$ edge-types of H_2 which are not yet present in Q, and $n - 3$ remaining spoke edges. The remaining edges of H_2 form three paths:

$$P_u = [m, m_u, \ldots, a, u_1]$$
$$P_v = [v_1, c, \ldots, m_v, m]$$
$$P_2 = [u_2, b, \ldots, d, v_2]$$

Where m_u and m_v are the vertices adjacent to m in H_2, u_2 (and v_2 respectively) are the neighbours of u (and v respectively), and where P_2 does not pass through m. Let $P_1 = [P_v, P_u]$. These paths are illustrated in Figure 2.5b. As u, v, m were selected to satisfy Claim 5, P_1 and P_2 are both of even length. As P_1 is even in length, P_u and P_v must have the same length parity.

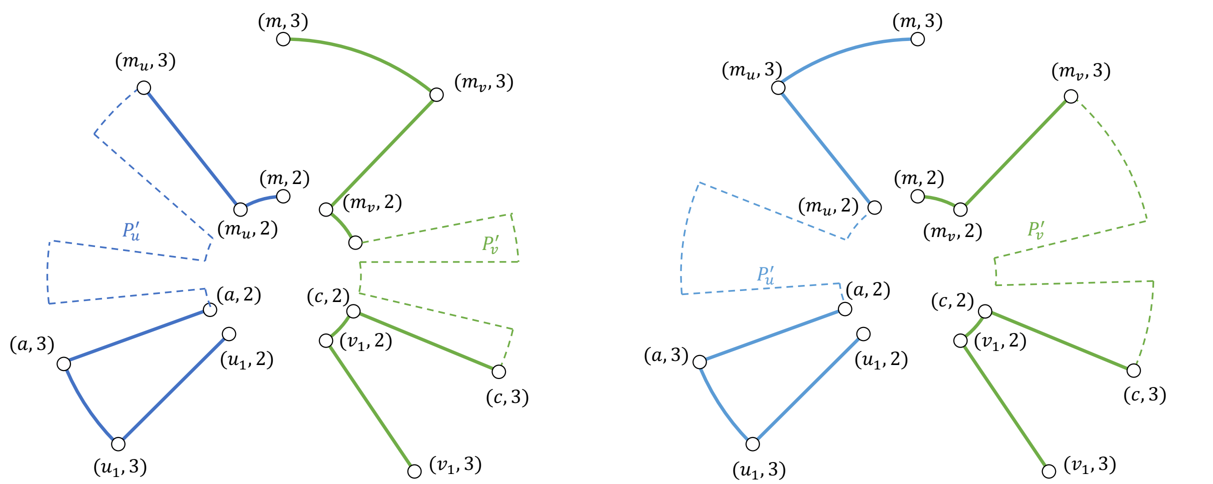
If both P_u and P_v are even (See Figure 2.6a):

$$P'_v = [(v_1, 3), (v_1, 2), (c, 2), (c, 3), \ldots, (m_v, 2), (m_v, 3), (m, 3)]$$

$$P'_u = [(u_1, 2), (u_1, 3), (a, 3), (a, 2), \ldots, (m_u, 3), (m_u, 2), (m, 2)]$$

$$P'_2 = [(u_2, 2), (u_2, 3), (b, 3), (b, 2), \ldots, (d, 3), (d, 2), (v_2, 2), (v_2, 3)]$$

$$Q = [R_1, \hat{P}'_u, F_u, P'_2, \hat{F}_v, P'_v, R_2]$$



(a) The case where the lengths of P_u and P_v are both even.

(b) The case where the lengths of P_u and P_v are both odd.

Figure 2.6: Illustration of how to construct P'_u and P'_v. P'_2 is omitted.

As in Sections 2.1 and 2.2, P'_u, P'_v and P'_2 alternate between levels 2 and 3. As both P_u and P_v are even, by alternating H_2 edges between level 2 and 3, connected by spoke edges, P'_u (P'_v respectively) starts in level 2 (level 3 respectively), visits each vertex in level 2 and 3 on the P_u (P_v respectively) path, ends in level 2 (level 3 respectively), and uses the same number of spoke and H_2 edges. Whereas P'_2 starts in level 2, visits each vertex in level 2 and 3 on the P_2 path, ends in level 3, and thus uses one more spoke edge than it does H_2 edge. Thus, each remaining H_2 and spoke edge has been used exactly once, and each vertex has been visited exactly once.

Otherwise, if both P_u and P_v are odd (See Figure 2.6b):

$$P'_v = [(v_1, 3), (v_1, 2), (c, 2), (c, 3), \ldots, (m_v, 3), (m_v, 2), (m, 2)]$$

$$P'_u = [(u_1, 2), (u_1, 3), (a, 3), (a, 2), \ldots, (m_u, 2), (m_u, 3), (m, 3)]$$

$$P'_2 = [(u_2, 2), (u_2, 3), (b, 3), (b, 2), \ldots, (d, 3), (d, 2), (v_2, 2), (v_2, 3)]$$

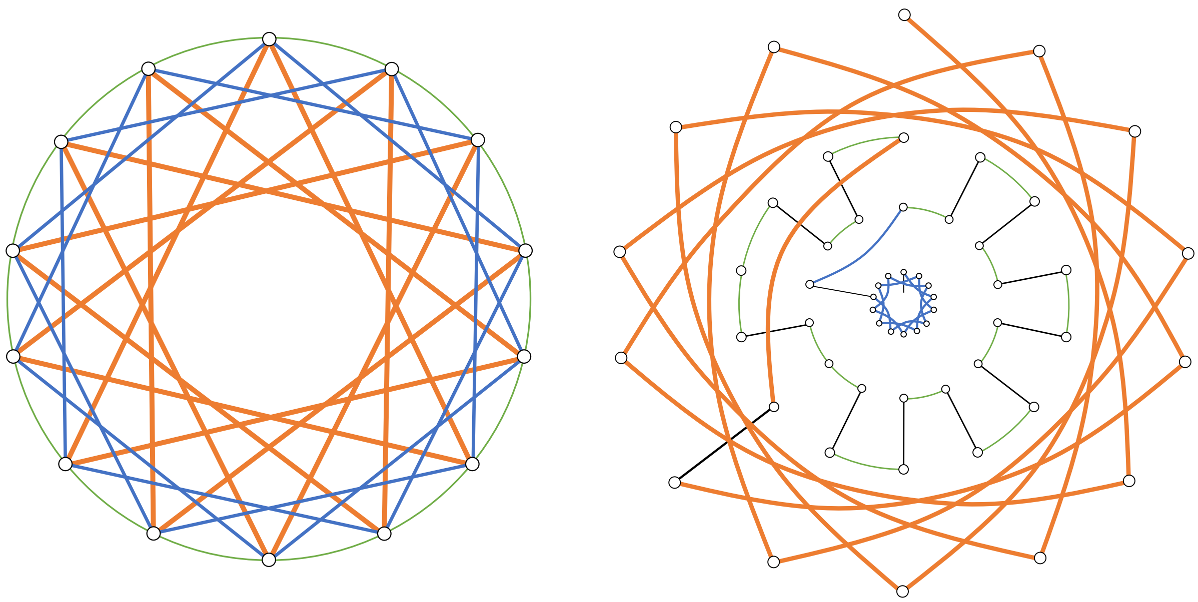
$$Q = [R_1, \hat{P}'_v, F_v, \hat{P}'_2, \hat{F}_u, P'_u, R_2]$$

As with the both even case, P'_u, P'_v and P'_2 alternate between levels 2 and 3. As both P_u and P_v are odd, by alternating H_2 edges between level 2 and 3, connected by spoke edges, P'_u (P'_v respectively) starts in level 2 (level 3 respectively), visits each vertex in level 2 and 3 on the P_u (P_v respectively) path, and ends in level 3 (level 2 respectively), using the same number of spoke and H_2 edges. Whereas P'_2 starts in level 2, visits each vertex in level 2 and 3 on the P_2 path, ends in level 3, and thus uses one more spoke edge than H_2 edge. Thus, each remaining H_2 edge-type and spoke edge has been used exactly once, and each vertex has been visited exactly once.

In both cases, Q visits every vertex in the 4-section exactly once, and uses each type of edge exactly once, so $Q_0 = \bigcup_{i \in \mathbb{Z}} Q + 4i, Q_1 = Q_0 + 1, Q_2 = Q_0 + 2, Q_3 = Q_0 + 3$ gives a Hamilton decomposition of $G \square P_\infty$ into double rays. $\square$

Note that while the proof of Claim 5 is not explicitly constructive, it does provide a heuristic search method for finding a suitable u, v, m triple, and they are not typically difficult to find. Additionally, the claim currently relies on G being simple, not a multigraph, but with some alteration this assumption should be avoidable. An example construction for $Cay((\mathbb{Z}_{14}, +), \{1, 3, 5\}) \square P_\infty$ is provided in Figure 2.7.

Note that as with Section 2.2, we can combine Claim 3, which provides a connection between Cayley graphs and Cartesian products, and Theorem 7 to provide a Hamilton decomposition for a large number of 8-regular, two-ended Cayley graphs with exactly one non-torsion generating element. If Alspach's Conjecture for finite graphs were true, then it would be a complete result for such graphs, but as this remains unsettled, and the exemption of multi-graphs remains, this is not a complete result.



(a) $Cay((\mathbb{Z}_{14}, +), \{1, 3, 5\})$. Each colour represents one cycle. H_1 is blue, H_2 is green and thin, H_3 is orange and thick.

(b) The canonical path Q for $Cay((\mathbb{Z}_{14}, +), \{1, 3, 5\}) \square P_\infty$ with the colours of the original decomposition retained for clarity.

Figure 2.7: An example of extending a Hamilton decomposition of $Cay((\mathbb{Z}_{14}, +), \{1, 3, 5\})$ to Hamilton decomposition of its infinite prism.

Theorems 5 and 7 suggest the following conjecture for higher valency base graphs.

Conjecture 5. *If G is a connected $2(k-1)$-regular graph that admits a Hamilton decomposition into $k-1$ cycles, and $|G| \equiv k \pmod 2$, then $G \square P_\infty$ admits a Hamilton decompositon into k double-rays.*

Even for 8-regular base graphs, the techniques applied here appear to be infeasible, as the number of special vertices increases significantly. Very little evidence has been gathered to test this conjecture for 8-regular base graphs or above. In the next section we summarise why the techniques applied so-far appear to be infeasible.

2.4 8-valent base graphs

Let G be an 8-regular graph with a Hamilton decomposition into four cycles H_1, H_2, H_3, H_4, with edges coloured blue, orange, green and grey respectively. The techniques from the previous cases have involved placing a pair of Hamilton paths ($H_1 - e$ and $H_2 - e'$) in the highest and lowest levels of a k-section, and placing the remaining edges e and e' in the neighbouring levels connected with spokes. If we do that in the 8-valent base graph case (see Figure 2.8) we can count the number of loose ends left to be tied together with zigzagging. The spoke edges used to connect e and e' to their respective Hamilton paths now force two vertices each to be visited without the use of a spoke edge ($(2,u),(3,u)$ and $(2,v),(3,v)$). This leaves eight loose ends, plus the loose ends at $(2,m)$ and $(4,m)$ from the other ends of e and e', plus the edges which must visit $(3,m)$ without using a spoke edge making twelve total loose ends. Worst of all is that there is no guarantee that these loose ends are even unique. K_9 is an 8-regular base graph with a Hamilton decomposition into four cycles, and yet it only has nine vertices, so there is certainly overlap in the loose ends presented in Figure 2.8.

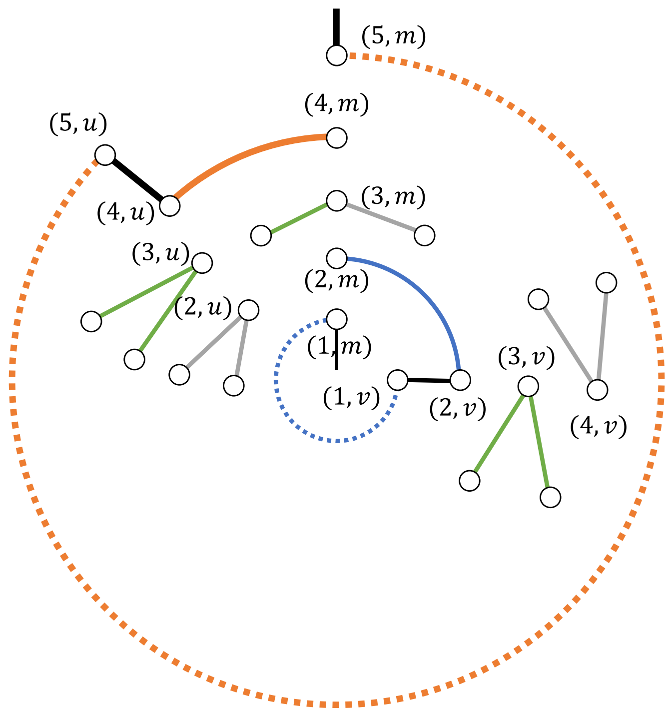


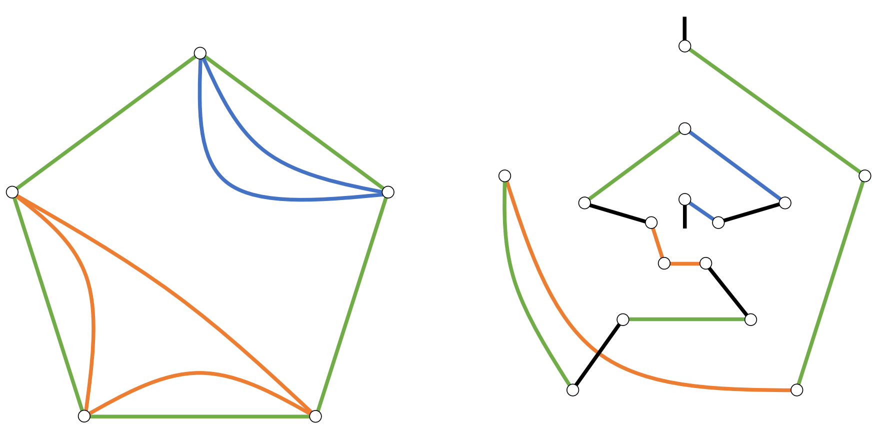
Figure 2.8: The special vertices that need to be connected together in the 8-regular base graph case to form a canonical path using the techniques established in this chapter.

Chapter 3

Hamilton decompositions of $G\Box P_\infty$ where G is not Hamilton decomposable.

We have seen that for $G\Box P_\infty$ to be Hamilton decomposable into two, three, or four double-rays, it is sufficient for the base graph G to have a Hamilton decomposition. In the 2-regular base graph case this condition is trivially necessary. In this chapter, we show that this condition is not necessary in the 4-regular base graph case, and provide an infinite family of solutions to support the conjecture that it is sufficient for the base graph to be Hamiltonian.

3.1 Hamiltonian base graphs



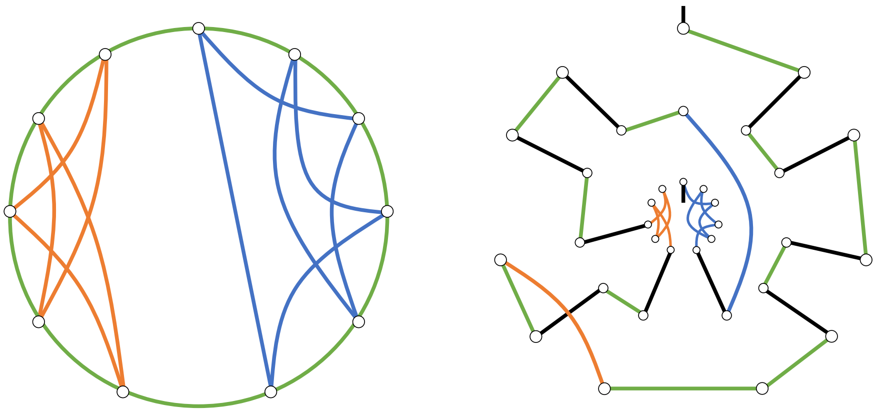
(a) A graph D_1 with a Hamilton cycle (green edges), but no Hamilton decomposition (only 2-edge-connected).

(b) The canonical path of $D_1\Box P_\infty$, with the colours of the original decomposition retained for clarity.

Figure 3.1: A small 4-regular Hamiltonian graph with no Hamilton decomposition, but whose infinite prism does have a Hamilton decomposition.

The results in Figures 3.1, 3.2, and 3.3 immediately demonstrate by example of canonical paths, that a Hamilton decomposition of G is not necessary for $G\Box P_\infty$ to be Hamilton decomposable. A stronger conjecture than Conjecture 5, which may be a better infinite generalisation of Alspach and Rosenfeld's Conjecture is offered here to motivate this chapter:

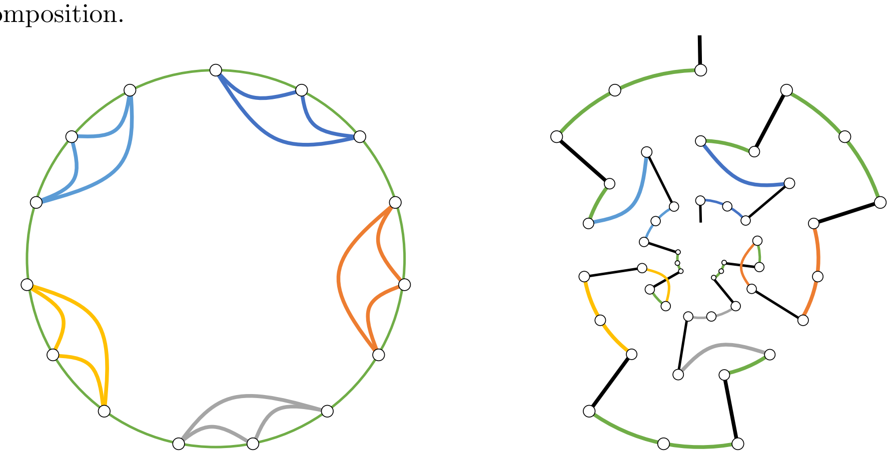
Conjecture 6. *If G is a connected $2(k-1)$-regular Hamiltonian graph, and $|G| \equiv k \pmod 2$, then $G \Box P_\infty$ admits a Hamilton decomposition into k double-rays.*



(a) A graph D_2 with a Hamilton cycle (green edges), but no Hamilton decomposition (only 2-edge-connected).

(b) The canonical path of $D_2 \Box P_\infty$, with the colours of the base graph decomposition retained for clarity.

Figure 3.2: A 4-regular Hamiltonian graph D_2 with no Hamilton decomposition, but $D_2 \Box P_\infty$ does have a Hamilton decomposition.



(a) A graph D_3 with a Hamilton cycle (green edges), but no Hamilton decomposition.

(b) The canonical path of $D_3 \Box P_\infty$, with the colours of the base graph decomposition retained for clarity.

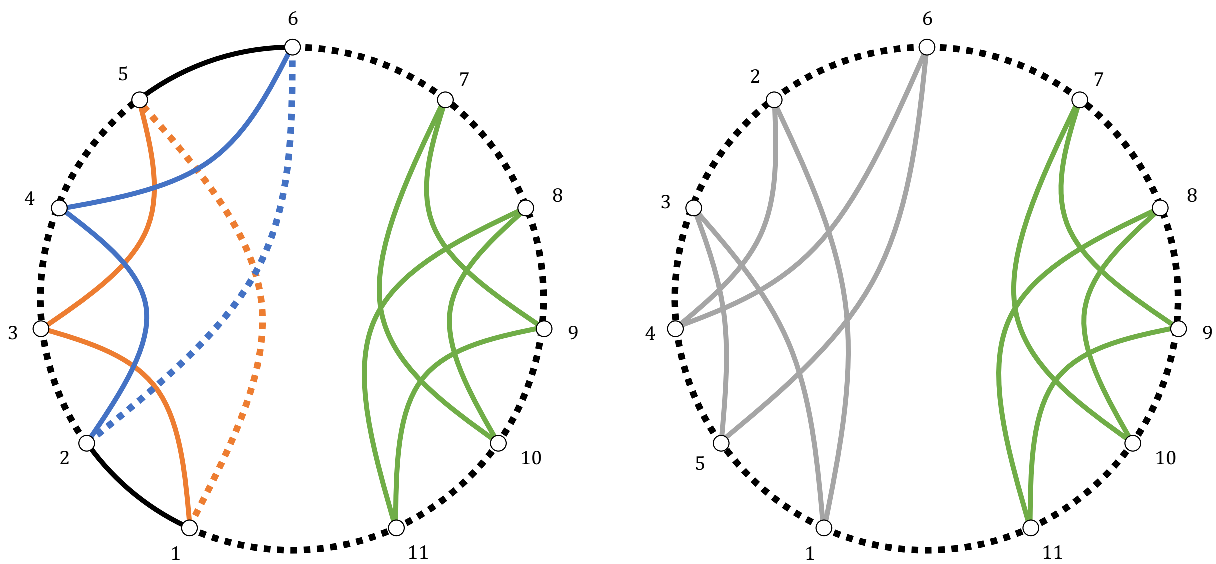
Figure 3.3: A 4-regular Hamiltonian graph D_3 with no Hamilton decomposition, but $D_3 \Box P_\infty$ does have a Hamilton decomposition. Note that the canonical path is not spanning on a 3-section, but it is nonetheless a valid solution on a 3-finite reduction.

The examples shown in Figures 3.1, 3.2, and 3.3 demonstrate a heuristic process via which $G \Box P_\infty$ may be Hamilton-decomposed if G is Hamiltonian. If G is a 4-regular graph with a Hamilton cycle H, let F be the subgraph such that $E(F) = E(G) \setminus E(H)$. F will be a 2-regular subgraph. If F is connected, then it is another Hamilton cycle, so G has a Hamilton decomposition, and this was already considered in Section 2.2. Otherwise, F is a collection of smaller cycles $C^1, \ldots, C^\ell$. For the remainder of this section, unless otherwise specified, *a smaller cycle* refers to a connected component C^i of F. A *smaller path* R^i is a path in $G \Box P_\infty$ consisting of all the edge-types from C^i, and some of the edge-types of H. If a collection of R^i can be connected together to form a path Q which spans a 3-finite-reduction and uses each edge-type of H exactly once, then Q is a canonical path in $G \Box P_\infty$.

We generalise this pattern with the following construction, which provides an infinite family of examples in support of Conjecture 6 in the 4-valent base-graph case.

Theorem 8. *Let G be a 4-regular Hamiltonian graph of odd order, with Hamilton cycle H. Let $C^1, \ldots, C^\ell$ be the cycles formed by $E(G) \setminus E(H)$. If for every smaller cycle C^i, $V(C^i)$ is a set of consecutive vertices on H, then $G \square P_\infty$ has a Hamilton decomposition into double-rays.*

For a 4-regular Hamiltonian graph G, if $V(C^i)$ is a set of consecutive vertices on H for every smaller cycle C^i, we will call G disentangled, and if there is a cycle C^j whose vertices are not consecutive vertices on H, we will call the graph G and the smaller cycle C^j entangled. Note that G is only entangled or disentangled with respect to a particular Hamilton cycle. (See Figure 3.4) One may be able to redraw an entangled graph as a disentangled graph by switching some edges to find a new Hamilton cycle. The particular edge-switching technique used in Figure 3.4 has been used extensively in the literature for combining cycles.



(a) A 4-regular graph with a Hamilton cycle in black, and three smaller cycles in green, blue and orange respectively. The blue and orange smaller cycles are currently entangled, but a new Hamilton cycle (dotted edges) can be used to disentangle G.

(b) The same 4-regular graph drawn with respect to the new Hamilton cycle (dotted edges). The graph now has two disentangled cycles in green and grey.

Figure 3.4: An illustration of the untangling process.

The restriction to disentangled graphs may seem arbitrary, but it represents the most natural example of a 4-regular Hamiltonian graph without a Hamilton decomposition, that being a graph which is only 2-edge-connected. If a disentangled graph has more than one smaller cycle (the alternative being only one other cycle, which is necessarily another Hamilton cycle), then it is necessarily only 2-edge-connected. Examples of 4-regular graphs which are Hamiltonian, 4-edge-connected, but not Hamilton decomposable are rather sparse, and these are considered in Section 3.2. Note that an even-regular graph must be even-edge-connected, as an odd edge-cut would create connected components which have an odd number of odd-degree vertices, which is not possible.

The proof of Theorem 8 will be broken up into cases, based on the parity of the number ℓ of smaller cycles. For a given smaller cycle C^i, let G^i be the subgraph of G induced by $V(C^i)$. The aim is to build smaller paths R_i on a 3-section of each $G^i \square P_\infty$ in such a way that they can be concatenated together to form a canonical path in $G \square P_\infty$. Note that this canonical path will no longer be built on a 3-section as in sections 2.1, 2.2 and 2.3, but they will be a solution in a 3-finite reduction, as can be seen in Figure 3.3. The canonical path requires a net vertical displacement of 3, but each R_i will have some displacement in order to span its respective $G^i \Box P_\infty$ 3-section. The strategy taken in the following construction is to have every cycle of a certain length parity have an R_i of the same net vertical displacement. Then certain R_i can be *vertically inverted* so that the net vertical displacement of various paths cancel out, so that the net vertical displacement of the canonical path is still 3. By *vertically invert*, we mean that by replacing each spoke edge $\{(x, y), (x, y+1)\}$ with the spoke edge $\{(x, y), (x, y-1)\}$ (*inverting* each spoke edge) the vertical displacement of the path will become negative.

Throughout the following constructions, we will label the vertices of G from 0 to $n-1$ with respect to the natural labelling given by the Hamilton cycle H, where $n = |V(G)|$. When it is said that two vertices of C^i are adjacent, it is meant that they are adjacent through an edge not in H, unless otherwise specified. We will also define the *left-most* vertex in C^i by $u_i \in V(C^i)$ such that $u_i - 1 \notin V(C^i)$. Similarly, we will define the *right-most* vertex in C^i by $v_i \in V(C^i)$ such that $v_i + 1 \notin V(C^i)$. Further for two vertices $x, y \in V(C^i)$, we will say that x *is to the left of* y if and only if $x, x+1, \ldots, y-1, y \in V(C^i)$. For an edge xy, where x is to the left of y in C^i, let $(y-x)$ be the length of xy, and let the parity of $(y-x)$ be the parity of the xy-edge.

Throughout the following constructions, illustrations will be presented in the Cartesian format as per Figure 1.5d, and vertices will not be labelled individually, but rows and columns will be labelled above and to the right of the figure. Dotted lines will be used to represent a path which visits a set of specific vertices, but where the order of vertices visited is not specified. Finally, some preliminary conclusions based on the parity of ℓ. As G must be of odd order, the number of odd-length cycles must be odd, and thus the number of even-length cycles must have opposite parity to the total number of cycles.

3.1.1 Building the path segments

The strategy is much the same for each of the cases. Place all of the C^i edges as a path P_1 in level 0, except for one edge xy raised into level 1, which has a specific parity. Then, using the H and spoke edges, connect the ends of P_1 to both sides of the $V(C^i)$ induced subgraph of G in a specific pair of levels to ensure the desired net vertical translation, using the parity of the number of vertices, and the C^i edge in level 1 to ensure that the combined path spans a 3-finite reduction. P_x will be the path connected to the end of P_1 in the direction of the x vertex, and P_2 will be the path connecting the other end of P_1. The main obstruction is that for the raised edge xy, the H edges of type $\{x-1, x\}$ and $\{x, x+1\}$ must be placed in level 2 or -1, directly above or below the spoke edge $\{(x, 0), (x, 1)\}$, as if they were placed in level 3 (or -2 respectively), translating this path vertically by 3 would cause a self-intersection at $(x, 0)$ (or $(x, 1)$ respectively). Although P_1, P_2 and P_x will be dependent on the C^i from which they are built, they will not be indexed as such, as they are only *helper* paths (in the programming sense) used to build R_i, and thus will not be referred to outside of the specific C^i in question.

Case 1: C^i odd

See Figure 3.5 for an illustration of the construction. For each odd cycle C^i, the aim is to build a path R_i which starts at (u_i, a) and ends at $(u_{i+1}, a+3)$, using exactly one of each edge type along the way. This can later be vertically inverted to be a path from (u_i, a) to $(u_{i+1}, a-3)$ if the overall construction requires it. Let xy be an odd-length edge in C^i. Such an edge is guaranteed as otherwise C^i would have only even-length edges, and thus would not be connected. Let $C^i = (x, y, z_1, z_2, \dots, z_{t_i-2})$, where t_i is the length of C^i.

$$\begin{aligned}
P_1 =& [(y,-1),(y,0),(z_1,0),(z_2,0),\dots,(z_{t_i-3},0),(z_{t_i-2},0),(x,0),(x,1),(y,1)] \\
P_x =& [(u_i,-2),(u_i,-1),(u_i+1,-1),(u_i+1,-2),\dots, \\
& (x-1,-2),(x-1,-1),(x,-1),(x+1,-1),(x+1,-2),\dots, \\
& (y-1,-2),(y-1,-1),(y,-1)] \\
P_2 =& [(y,1),(y+1,1),(y+1,2),\dots,(v_i-1,1),(v_i-1,2),(v_i,2),(v_i,1),(u_{i+1},1)] \\
R_i =& [P_x, P_1, P_2]
\end{aligned}$$

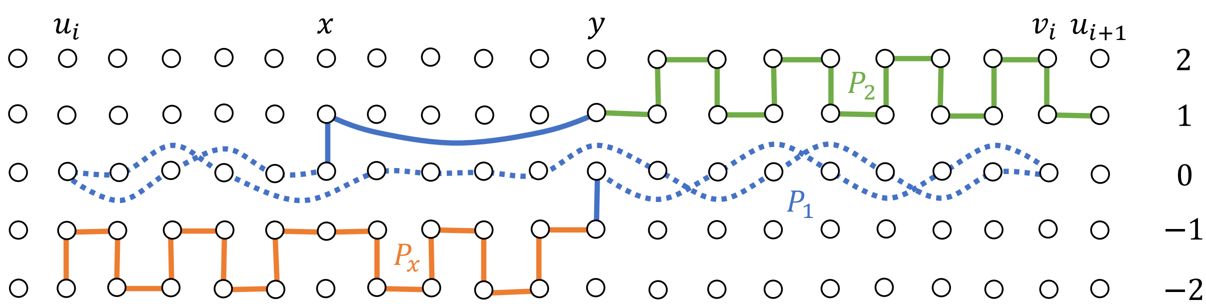


Figure 3.5: R_i when C^i is of odd length. The dotted lines of P_1 indicate that the edges are not specific, but note that as P_1 is constructed from the edges of C^i, P_1 does visist every vertex in level 0 between u_i and v_i in some order.

P_1 is all of the edges from C^i placed into level 0, except for the xy edge which is raised into level 1. It uses the x and y spoke edges. P_x is all of the edges from H between u_i and y, alternated between levels -1 and -2. As the x-spoke edge is already used by P_1, this spoke edge is not used, and as xy is an odd edge, there are an even number of vertices (and therefore spokes and level changes) between x and y, so the $\{x-1, x\}$ and $\{x, x+1\}$ edges will always be placed in level -1. This also ensures that P_x and P_1 together span a 3-finite reduction to the left of y. As there are no spoke edges used by P_1 to the right of y, P_2 can use all the remaining spoke edges on the right of y, and thus P_1 and P_2 span a 3-finite reduction to the right of y, so R_i spans a 3-finite reduction between u_i and v_i. Note that in the given construction, it is assumed that $x - u_i$ is odd, and $v_i - y$ even. In general, it may be the case that $x - u_i$ is even, and $v_i - y$ odd. However, the only change is that P_x will start in level -1 instead of -2, and P_2 will finish in level 2 instead of level 1. In either case, the overall vertical displacement of R_i is still 3.

Case 2: C^i even

For each even cycle C^i, the aim is to build a path R_i which starts at (u_i, a) and ends at $(u_{i+1}, a+2)$. Again, this can later be vertically inverted to be a path from (u_i, a) to $(u_{i+1}, a-2)$ if the overall construction requires it. The construction of this case requires an edge with certain conditions. An edge with these conditions is guaranteed by the following lemma:

Lemma 4. *For an even-length cycle C^i of length n, where the vertices are labelled as consecutive integers from 0 to $n-1$ (not necessarily according to C^i), there exists an edge xy, with $x < y$, that satisfies one of these conditions:*

- *$y - x$ is even, and x is odd,*
- *$y - x$ is odd, and x is even.*

Proof. Let $y = n-1$, and note that each edge $\{x, n-1\} \in C^i$ has $x < n-1$. As n is even, if x is even, then $(n-1) - x$ is odd, and if x is odd, then $(n-1) - x$ is even. In either case the conditions are satisfied. $\square$

If, for the vertices of a particular C^i, we temporarily label each vertex by the number of vertices in $V(C^i)$ to its left, we can apply Lemma 4 to guarantee that there always exists an edge xy such that one of the following two cases applies. Note that the proof of Lemma 4 shows that an edge of the form $\{x, n-1\}$ always satisfies the conditions, but for completeness, the proof is provided for any xy edge which satisfies the conditions.

Case 2.1: xy an even edge with an odd number of vertices to the left of x, and $x < y$.

See Figure 3.6 for an illustration of the construction. Let xy be an even edge with an odd number of vertices to the left of x, and $x < y$. Let $C^i = (x, y, z_1, z_2, \ldots, z_{t_i-2})$. The construction of R_i is then as follows:

$$\begin{aligned}
P_1 =&[(y,-1),(y,0),(z_1,0),(z_2,0),\ldots,(z_{t_i-3},0),(z_{t_i-2},0),(x,0),(x,1),(y,1)]\\
P_x =&[(u_i,1),(u_i,2),(u_i+1,2),(u_i+1,1),\ldots,\\
&(x-1,1),(x-1,2),(x,2),(x+1,2),(x+1,1),\ldots,\\
&(y-1,2),(y-1,1),(y,1)]\\
P_2 =&[(y,-1),(y+1,-1),(y+1,-2),\ldots,(v_i-1,-1),(v_i-1,-2),(v_i,-2),(v_i,-1),(u_{i+1},-1)]\\
R_i =&[P_x,\hat{P}_1,P_2]
\end{aligned}$$

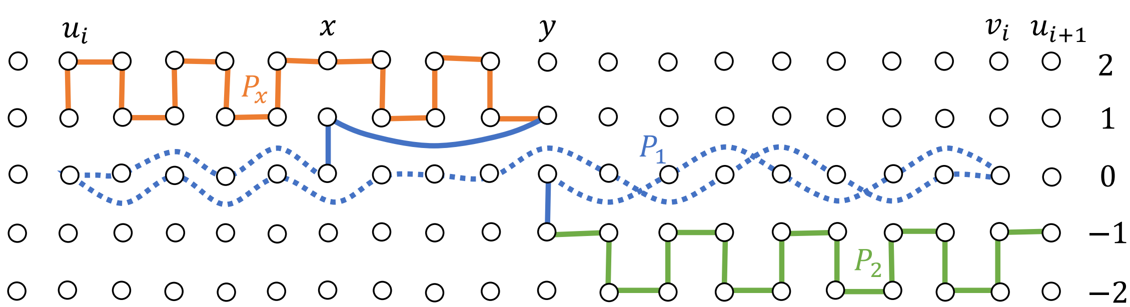


Figure 3.6: R_i when C^i is of even length and xy is even. The dotted lines of P_1 indicate that the edges are not specific, but note that as P_1 is constructed from the edges of C^i, P_1 does visist every vertex in level 0 between u_i and v_i in some order.

P_1 is again all the edges of C^i placed in level 0, except for the xy edge which is raised into level 1. As xy is even, there are an odd number of vertices (and therefore spokes) between y and x, so P_x can alternate between levels 1 and 2 and the edges of type $\{x-1, x\}$ and $\{x, x+1\}$ will always end up in level 2. Further, as the number of vertices to the left of x is odd, it will always start in level 1 and connect up to the $\{x-1, x\}$ and $\{x, x+1\}$ edges in level 2. On the other side, P_2 alternates between levels -1 and -2 using every spoke edge, and as $y - x$ is even, C^i is even, and there are an odd number of vertices to the left of x, there are an even number of vertices to the right of y, so P_2 uses an even number of spoke edges, so ends in level -1. Thus, the net vertical displacement of R_i is 2.

Case 2.2: xy an odd edge with an even number of vertices to the left of x, and $x < y$.

See Figure 3.7 for an illustration of the construction. Let xy be an odd edge with an even number of vertices to the left of x, and $x < y$. Let $C^i = (x, y, z_1, z_2, \ldots, z_{t_i-2})$. The construction of R_i is then as follows:

$$\begin{aligned}
P_1 =& [(y,-1),(y,0),(z_1,0),(z_2,0),\ldots,(z_{t_i-3},0),(z_{t_i-2},0),(x,0),(x,1),(y,1)]\\
P_x =& [(u_i,-1),(u_i,-2),(u_i+1,-2),(u_i+1,-1),\ldots,\\
& (x-1,-2),(x-1,-1),(x,-1),(x+1,-1),(x+1,-2),\ldots,\\
& (y-1,-2),(y-1,-1),(y,-1)]\\
P_2 =& [(y,1),(y+1,1),(y+1,2),\ldots,(v_i-1,1),(v_i-1,2),(v_i,2),(v_i,1),(u_{i+1},1)]\\
R_i =& [P_x,P_1,P_2]
\end{aligned}$$

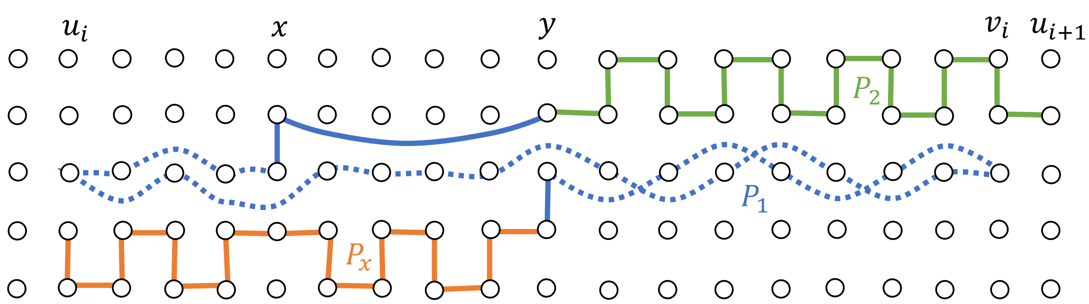
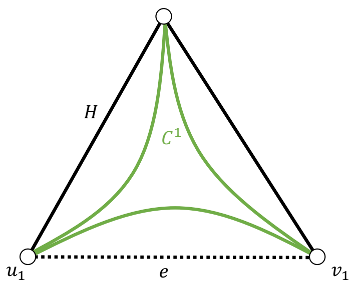


Figure 3.7: R_i when C^i is of even length and xy is odd. The dotted lines of P_1 indicate that the edges are not specific, but note that as P_1 is constructed from the edges of C^i, P_1 does visist every vertex in level 0 between u_i and v_i in some order.

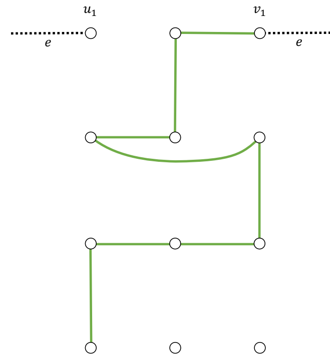
Again, P_1 is all the edges of C^i placed in level 0, except for the xy edge which is raised to level 1. As xy is odd, there are an even number of vertices (and therefore spokes) between y and x, so P_x can alternate between levels -1 and -2 and the edge-types $\{x-1, x\}$ and $\{x, x+1\}$ will always end up in level -1. On the other side, P_2 alternates between levels 1 and 2, using every spoke edge, and as $y - x$ is odd, and there are an even number of vertices to the left of x, there are an even number of vertices to the right of y, so P_2 uses an even number of spoke edges, so ends in level -1. Thus, the net vertical displacement of R_i is $+2$.

3.1.2 Gluing the path segments together

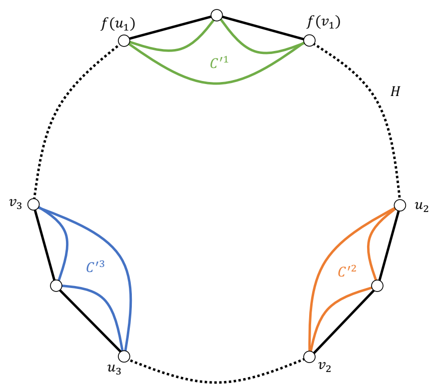
The above constructions provide a strategy for inserting a pair of cycles with the same parity into an already existing decomposition of $G \square P_\infty$. An example of a pair of graphs where two cycles can be inserted into G to create G' can be seen in Figure 3.8a and Figure 3.8c.



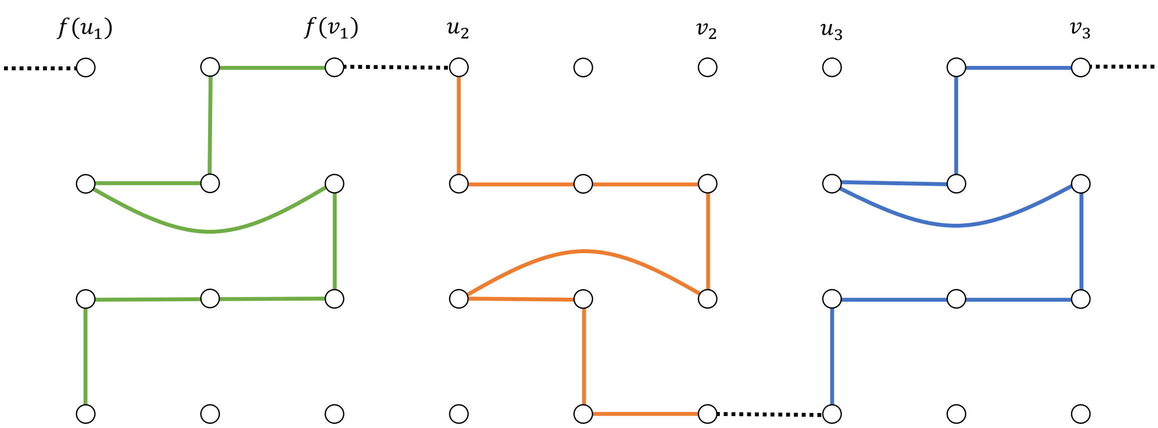
(a) $G = 2K_3$. Note that for this graph, $e_a = e_b = e$.



(b) The canonical path of $G \square P_\infty$. The two dotted lines labelled e are the same type-e edge, wrapped around the figure in this representation.



(c) G', the graph G with two cycles C'^2 and C'^3 inserted.



(d) The canonical path of $G' \square P_\infty$. The orange and blue paths are constructed from Section 3.1.1.

Figure 3.8

Claim 6. *Let G and G' be a pair of 4-regular graphs such that:*

- *G and G' each have a Hamilton cycle, H and H' respectively,*
- *$E(G) \setminus E(H)$ forms a sequence of $(\ell - 2)$ cycles*

$$C^1, C^2, C^3, \ldots, C^{a-1}, C^{a+1}, \ldots, C^{b-1}, C^{b+1}, \ldots, C^{\ell-1}, C^{\ell},$$

where each $V(C^i)$ is a set of consecutive vertices on H, and $\{v_i, u_{i+1}\} \in E(H)$, where v_i is the right-most vertex of C^i and u_{i+1} is the left-most vertex of C^{i+1} (except when $i = a, b$ in which case $\{v_{i-1}, u_{i+1}\} \in E(H)$),

- *$E(G') \setminus E(H')$ forms a sequence of ℓ cycles*

$$C'^1, C'^2, C'^3, \ldots, C'^{a-1}, C'^a, C'^{a+1}, \ldots, C'^{b-1}, C'^b, C'^{b+1}, \ldots, C'^{\ell-1}, C'^{\ell},$$

where each $V(C'^i)$ is a set of consecutive vertices on H, and $\{v'_i, u'_{i+1}\} \in E(H')$, where v'_i is the right-most vertex of C'^i and u'_{i+1} is the left-most vertex of C'^{i+1}.

- *the subgraph of G' induced by*

  $V(C'^1) \cup V(C'^2) \cup V(C'^3) \cup \cdots \cup V(C'^{a-1}) \cup V(C'^{a+1}) \cup \cdots \cup V(C'^{b-1}) \cup V(C'^{b+1}) \cup \cdots \cup V(C'^{\ell-1}) \cup V(C'^{\ell})$

  is isomorphic to G, with two edges of H removed:

  – *e_a, an edge of H with one end in C^{a-1} and one end in C^{a+1},*
  – *e_b, an edge of H with one end in C^{b-1} and one end in C^{b+1},*

- *C'^a and C'^b have the same parity.*

Then $G \square P_\infty$ has a Hamilton decomposition into translation-equivalent double-rays of period 3 implies that $G' \square P_\infty$ has a Hamilton decomposition into translation-equivalent double-rays of period 3.

Proof. Assume that $G \square P_\infty$ has a Hamilton decomposition into translation-equivalent double-rays of period 3, and show that $G' \square P_\infty$ does too. Let Q_0, Q_1, Q_2 be the Hamilton double-rays which decompose $G \square P_\infty$, and let Q be one period of Q_0. As the other paths are equivalent under translation, it is sufficient to show that there exists a canonical path Q' in $G' \square P_\infty$.

Let $e_a = \{v_{a-1}, u_{a+1}\}$ be the edge of H with one end $v_{a-1} \in V(C^{a-1})$ and the other end $u_{a+1} \in V(C^{a+1})$. Similarly, let $e_b = \{v_{b-1}, u_{b+1}\}$ be the edge of H with one end $v_{b-1} \in V(C^{b-1})$ and the other end $u_{b+1} \in V(C^{b+1})$. Let $\{(v_{a-1}, c), (u_{a+1}, c)\} \in E(Q)$ be the edge of type e_a in Q, and let $\{(v_{b-1}, d), (u_{b+1}, d)\} \in E(Q)$ be the edge of type e_b in Q. Let $f : V(G) \mapsto V(G')$ be the isomorphism between $G \setminus \{e_a, e_b\}$ and the subgraph of G' induced by $V(C'^1) \cup \cdots \cup V(C'^{a-1}) \cup V(C'^{a+1}) \cup \cdots \cup V(C'^{b-1}) \cup V(C'^{b+1}) \cup \cdots \cup V(C'^{\ell})$. Note that f induces an isomorphism f^* from $G \square P_\infty$ to $G' \square P_\infty$, where $f^*((x, y)) = (f(x), y)$. Let $P = Q \setminus \{\{(v_{a-1}, c), (u_{a+1}, c)\}, \{(v_{b-1}, d), (u_{b+1}, d)\}\}$ be the canonical path Q with the e_a and e_b type edges removed. Without loss of generality, let the e_a-type edge appear before the e_b-type edge in P. Note that P is now split into three paths P_1, P_2, P_3. Let P_1 be the first path, which will end at vertex (v_{a-1}, c), P_2 the second path from (u_{a+1}, c) to (v_{b-1}, d), and P_3 the path from (u_{b+1}, d). Let P'_1, P'_2, and P'_3 be the images of P_1, P_2 and P_3 under the isomorphism f^*. The aim is to find a path which connects $(f(v_{a-1}), c)$ to $(f(u_{b+1}), d)$ using P'_2 and the constructions provided for spanning a 3-finite reduction of $G'[V(C'^a)] \square P_\infty$ and $G'[V(C'^b)] \square P_\infty$ in Section 3.1.1. Let P'_a and P'_b be the segments constructed from C'^a and C'^b respectively. As C'^a and C'^b are of the same parity, their net vertical displacement is the same. Note that P'_2 already has a net vertical displacement of $(d - c)$, as it starts at $(f(u_{a+1}), c)$ and ends at $(f(v_{b-1}), d)$, so by vertically inverting one of P'_a or P'_b, and translating P'_2 by either 2 or 3 (depending on the parity of C'^a and C'^b) so that the ends of P'_2 meet P'_a and P'_b, the net vertical displacement of $[P'_a, P'_2, P'_b]$ is $(d - c)$. Thus, it connects $(f(v_{a-1}), a)$ to $(f(u_{b+1}), d)$, $Q' = [P'_1, P'_a, P'_2, P'_b, P'_3]$ is a canonical path in $G' \square P_\infty$, and $G' \square P_\infty$ has a Hamilton decomposition into translation-equivalent double-rays of period 3.

□

Every G' which satisfies the conditions of Theorem 8 can be constructed from a smaller graph G in a way that inductively satisfies Claim 6, so it suffices to show the base cases where $\ell = 1$ and $\ell = 2$.

Case 1: Odd number of cycles

The number of odd-length cycles is odd, as $|G|$ is odd by the admissibility criterion, so if there are an odd number of cycles overall, the number of even-length cycles must be even. Any such graph G' that satisfies the conditions of Theorem 8 can be constructed from an odd-order graph G with a Hamilton decomposition into two cycles, due to Claim 6. The remaining cycles can then always be added two at a time with matching parity, as there are an even number of even-length cycles, and an even number of odd-length cycles, excluding the first one which is already in G before any cycles are inserted. The base case of showing $G\Box P_\infty$ has a Hamilton decomposition is already proven in Section 2.2, so the case of an odd number of cycles is done.

Case 2: Even number of cycles

The number of odd-length cycles is odd, as $|G|$ is odd, so if there are an even number of cycles, the number of even-length cycles must be odd too. It thus suffices to show that $G\Box P_\infty$ has a Hamilton decomposition for a graph G which can be decomposed into a Hamilton cycle H, and two smaller cycles C^1 and C^2 which satisfy the conditions of the theorem and Claim 6, where C^1 is an odd-length cycle and C^2 is an even length cycle.

As above, let u_i be the left-most vertex of C^i, and let v_i be the right-most vertex of C^i. The following paths are constructed from the odd-length cycle $C^1 = (x_1, y_1, z_1, z_2, \dots, z_{t_1-2})$. Let x_1y_1 be an even-length edge in C^1. Considering the length and direction of the edges of C^1, they must sum to 0, as C^1 is a cycle. As 0 is even, the number of odd-length edges in C^1 must be even. But there are an odd number of edges overall, so there must be an odd number of even-length edges. Thus, such an x_1y_1 must exist. If the number of vertices to the left of x_1 is odd, then the construction is as follows: (See Figure 3.9 for an illustration)

$$\begin{aligned}
P_1 =&[(v_2,3),(u_1,3),(u_1,2),(u_1+1,2),(u_1+1,3),\dots,\\
&(x_1-1,3),(x_1-1,2),(x_1,2),(x_1,1)]\\
P_2 =&[(x_1,1),(z_{t_1-2},1),(z_{t_1-3},1),\dots,(z_2,1),(z_1,1),(y_1,1)]\\
P_3 =&[(y_1,1),(y_1,2),(y_1-1,2),(y_1-1,3),(y_1-2,3),(y_1-2,2),(y_1-3,2),(y_1-3,3),\dots,\\
&(x_1+2,3),(x_1+2,2),(x_1+1,2),(x_1+1,3),(x_1,3)]\\
P_4 =&[(x_1,3),(y_1,3),(y_1+1,3),(y_1+1,2),(y_1+2,2),(y_1+2,3),\dots,\\
&(v_1-2,3),(v_1-2,2),(v_1-1,2),(v_1-1,3),(v_1,3),(v_1,2),(u_2,2)]\\
R_1 =&[P_1,P_2,P_3,P_4]
\end{aligned}$$

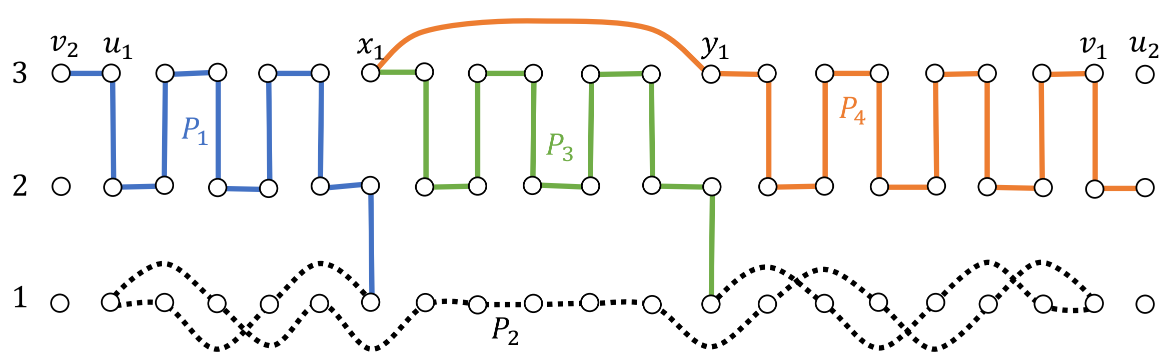


Figure 3.9: R_1, the path constructed from the odd-length cycle, when there is an even number of cycles, and there are an odd number of vertices to the left of x_1.

P_1 alternates edges of H between levels 2 and 3, from $(v_2, 3)$ to $(x_1, 1)$. Because there are an odd number of vertices to the left of x_1, there are an odd number of spokes, so the last H edge is always in level 2, so it can be connected to the x_1 spoke edge. P_2 is all of the edges of C^1 in level 1, except for the x_1y_1 edge. P_3 connects $(y_1, 1)$ to $(x_1, 3)$, alternating H edges between levels 2 and 3. As x_1y_1 is an even edge, there are an even number of spokes between y_1 and x_1 (including the y_1-spoke), so the net vertical displacement of $+2$ is always achieved. Then P_4 uses the remaining C^1 edge-type x_1y_1 and the remaining H and spoke edges to connect to $(u_2, 2)$. As there are an odd number of vertices to the right of y_1, P_4 is guaranteed to end in level 2.

If the number of vertices to the left of x_1 is even, then the construction is as follows: (See Figure 3.10 for an illustration)

$$\begin{aligned}
P_1 =&[(v_2,3),(u_1,3),(u_1,2),(u_1+1,2),(u_1+1,3),\dots,\\
&(x_1-1,2),(x_1-1,3),(x_1,3),(y_1,3)]\\
P_2 =&[(x_1,1),(z_{t_1-2},1),(z_{t_1-3},1),\dots,(z_2,1),(z_1,1),(y_1,1)]\\
P_3 =&[(y_1,3),(y_1-1,3),(y_1-1,2),(y_1-2,2),(y_1-2,3),(y_1-3,3),(y_1-3,2),\dots,\\
&(x_1+2,2),(x_1+2,3),(x_1+1,3),(x_1+1,2),(x_1,2),(x_1,1)]\\
P_4 =&[(y_1,1),(y_1,2),(y_1+1,2),(y_1+1,3),(y_1+2,3),\dots,\\
&(v_1-2,3),(v_1-2,2),(v_1-1,2),(v_1-1,3),(v_1,3),(v_1,2),(u_2,2)]\\
R_1 =&[P_1,P_3,P_2,P_4]
\end{aligned}$$

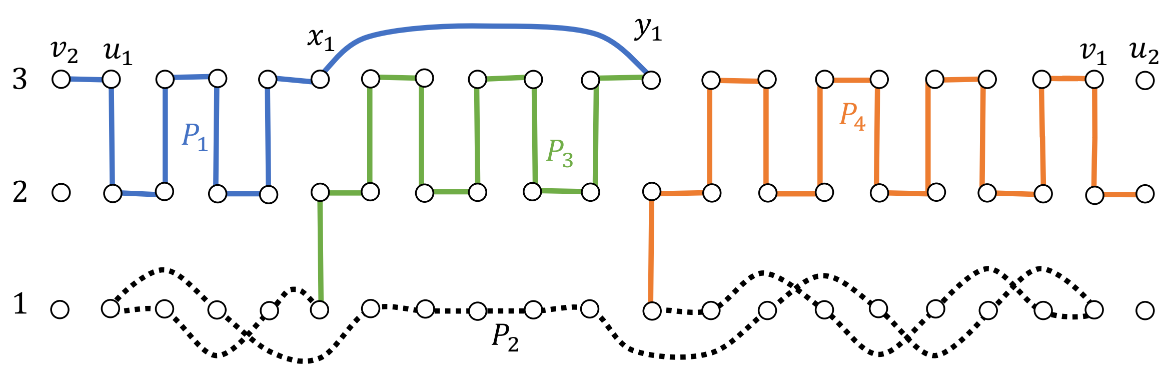


Figure 3.10: R_1, the path constructed from the odd-length cycle, when there is one even-length cycle and one odd-length cycle in the base case, and there are an even number of vertices to the left of x_1.

P_1 again uses all the H and spoke edges to the left of x_1, and the x_1y_1 edge-type from C^1. As the number of vertices to the left of x_1 is even, P_1 will start and end in level 3. P_3 then connects $(y_1, 3)$ to $(x_1, 1)$. As above, because of the even number of spokes between x_1 and y_1, including the x_1 spoke, P_3 will have a net vertical displacement of -2. P_2 again uses every C^1 edge except for the x_1y_1-type edge. Finally P_4 connects $(y_1, 1)$ to $(u_2, 2)$. As there are an odd number of spokes used, P_4 will always end in level 2.

In either case, R_1 is a path which connects $(v_2, 3)$ to $(u_2, 2)$, so with the remaining C^2 cycle, the aim is to build a canonical path from $(m, 0)$ to $(m, 3)$ for some $m \in V(C^2)$ which uses the R_1 path.

The even cycle C^2

Let $C^2 = (m, y, z_1, z_2, \ldots, z_{t_2-2})$. We will again make use of Lemma 4 to ensure that we have an edge my which is either an even edge with an odd number of vertices to the left of m, or an odd edge with an even number of vertices to the left of m.

In the case where my is an odd-length edge, we proceed as follows: (See Figure 3.11)

$$\begin{aligned}
S_1 =&[(m,0),(m,1),(z_{t_2-2},1),(z_{t_2-3},1),\ldots,(z_2,1),(z_1,1),(y,1),(y,2),(m,2)]\\
S_2 =&[(m,2),(m-1,2),(m-1,3),(m-2,3),(m-2,2),\ldots,\\
&(u_2+2,3),(u_2+2,2),(u_2+1,2),(u_2+1,3),(u_2,3),(u_2,2)]\\
S_3 =&[(v_2,3),(v_2,2),(v_2-1,2),(v_2-1,3),(v_2-2,3),(v_2-2,2),\ldots,\\
&(y+2,3),(y+2,2),(y+1,2),(y+1,3),(y,3),(y-1,3),(y-1,2),(y-2,2),(y-2,3),\ldots,\\
&(m+2,3),(m+2,2),(m+1,2),(m+1,3),(m,3)]\\
Q =&[S_1,S_2,\hat{R}_1,S_3]
\end{aligned}$$

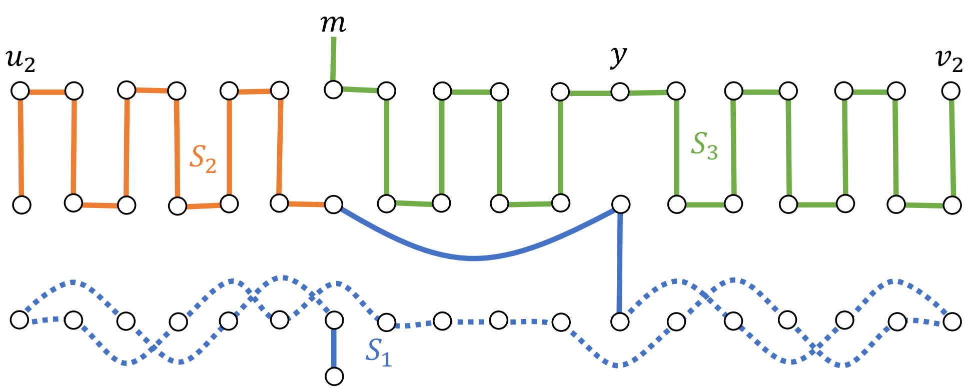


Figure 3.11: The parts of the canonical path constructed from the even-length cycle C^2, in the case where my is an odd-length edge and m has an even number of vertices to its left.

S_1 uses all the edges on C^2 in level 1, except for the my edge-type which is raised to level 2 and connected with a y-spoke edge. S_1 also connects $(m,0)$ to $(m,1)$. S_2 uses all of the H and spoke edges between u_2 and m. As there are an even number of vertices to the left of m, S_2 starts and ends in level 2. This connects it up with $(u_2,2)$, the start of $\hat{R}_1$. S_3 starts at $(v_2,3)$, where $\hat{R}_1$ ends, and uses all the H and spoke edges to the right of m, avoiding the $(y,2)$ vertex already used by S_1. As there are an even number of vertices to the right of y, S_3 passes over $(y,2)$ in level 3, and as there are an even number of vertices between m and y, it ends at $(m,3)$.

In the case where my is an even-length edge, the construction proceeds as follows: (See Figure 3.12)

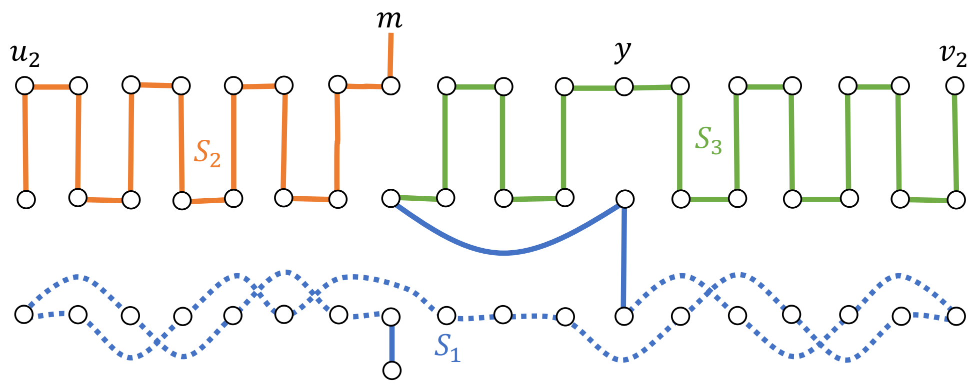


Figure 3.12: The parts of the canonical path constructed from the even-length cycle C^2, in the case where my is an even-length edge and m has an odd number of vertices to its left.

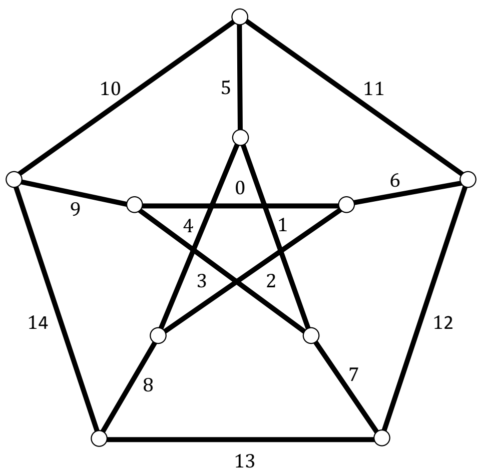
$$
\begin{aligned}
S_1 =& [(m,0),(m,1),(z_{t_2-2},1),(z_{t_2-3},1),\ldots,(z_2,1),(z_1,1),(y,1),(y,2),(m,2)] \\
S_2 =& [(m,3),(m-1,3),(m-1,2),(m-2,2),(m-2,3),\ldots, \\
& (u_2+2,3),(u_2+2,2),(u_2+1,2),(u_2+1,3),(u_2,3),(u_2,2)] \\
S_3 =& [(v_2,3),(v_2,2),(v_2-1,2),(v_2-1,3),(v_2-2,3),(v_2-2,2),\ldots, \\
& (y+2,3),(y+2,2),(y+1,2),(y+1,3),(y,3),(y-1,3),(y-1,2),(y-2,2),(y-2,3),\ldots, \\
& (m+2,2),(m+2,3),(m+1,3),(m+1,2),(m,2)] \\
Q =& [S_1,\hat{S_3},R_1,\hat{S_2}]
\end{aligned}
$$

Again, S_1 is all of the edges of C^2 in level 1, except for the my edge-type which is raised to level 2 and connected with a y-spoke edge. S_1 also connects $(m,0)$ to $(m,1)$. The only difference is that S_2 now connects to $(m,3)$ instead of $(m,2)$ (due to the odd number of vertices to the left of m), and S_3 now ends at $(m,2)$ (due to the odd number of vertices between m and y).

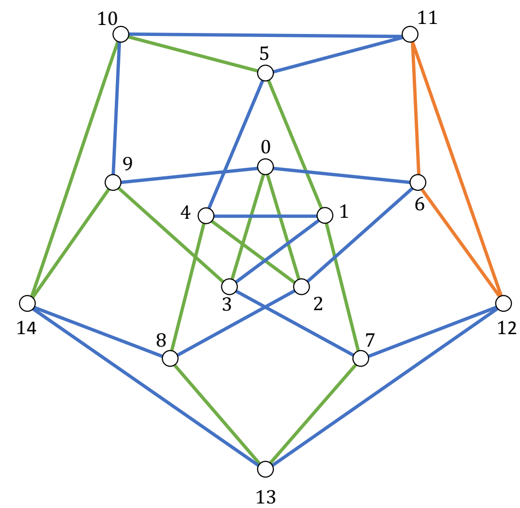
This completes the base cases required to use Claim 6 to build a Hamilton decomposition of any $G\Box P_\infty$, where G is Hamiltonian and the smaller cycles are consecutive vertices. This proves Theorem 8 and provides an infinite family of graphs that support Conjecture 6 in the 4-regular base graph case. The 4-regular base graph case is not solved, however. In general, the smaller cycles may be entangled with each other. Some examples where this is the case are demonstrated in the following section. Perhaps the best strategy for a general proof of Conjecture 6 in the 4-regular case is to show that with only a small number of base case exceptions, any 4-regular graph which is Hamiltonian but not Hamilton decomposable can be constructed by repeatedly inserting disentangled pairs of smaller cycles. There are various conjectures similar to this, and Bryant and Dean's 2015 paper [9] summarises some of these conjectures, but in general, not a lot is known.

3.2 Hamiltonian graphs with entangled smaller cycles

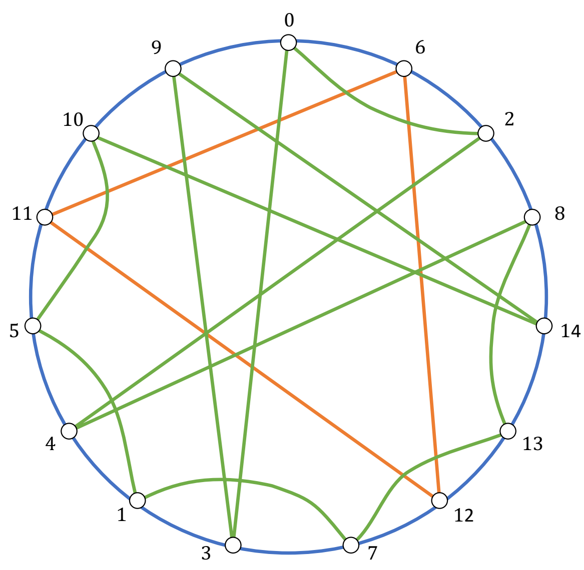
It is difficult to construct examples of 4-regular Hamiltonian graphs where the smaller cycles are entangled, but the graph does not have a Hamilton decomposition. However, one such graph can be derived from the Petersen graph P, together with the theorem of Kotzig, which states that a 3-valent graph has a Hamilton cycle if and only of its line graph has a Hamilton decomposition [24]. For a graph G, the line graph L_G is defined as the graph which has vertex set $V(L_G) = E(G)$, and two vertices of L_G are connected if and only if their corresponding edges in G are incident. The canonical path for a Hamilton decomposition of $L_P\Box P_\infty$ is shown in Figure 3.13, along with a few representations of P and L_P. One can apply the induction step from Claim 6 to the base case shown in Figure 3.13d to Hamilton decompose any graph which is L_P with other smaller cycles inserted.



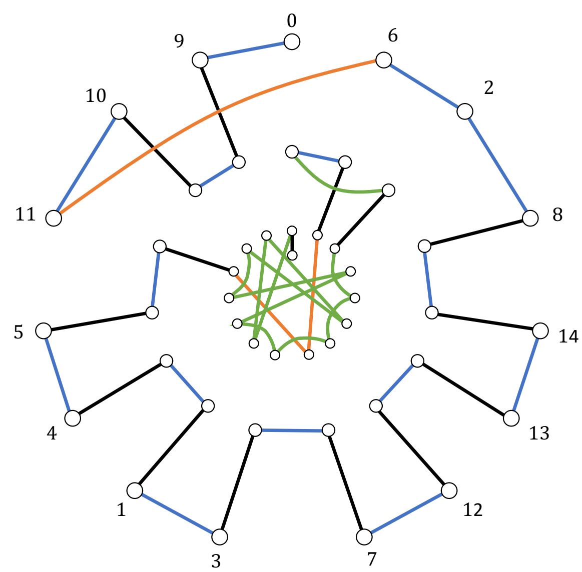
(a) The Petersen graph P.



(b) The line graph of the Petersen graph L_P. The Hamilton cycle is shown in blue, and the two smaller cycles shown in green and orange.



(c) L_P, redrawn in a circular orientation according to the Hamilton cycle.

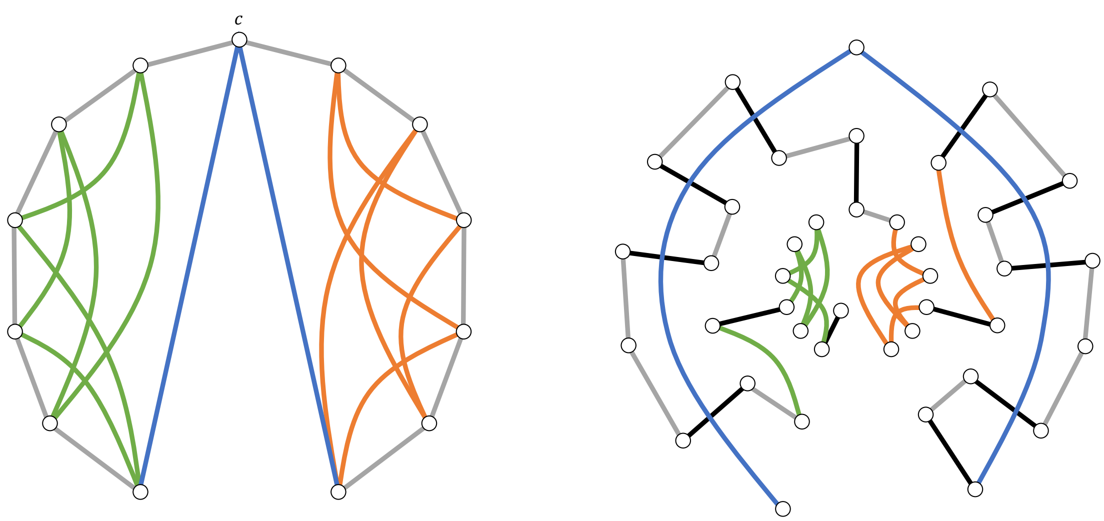


(d) The canonical path for an infinite decomposition of $L_P \square P_\infty$.

Figure 3.13: An example of an entangled 4-regular Hamiltonian graph where $G \square P_\infty$ is still Hamilton decomposable.

3.3 Semi-Hamiltonian base graphs

Alspach and Rosenfeld [4] established that if G is a cubic graph with a perfect 1-factorisation, then $G\square K_2$ has a Hamilton decomposition. They conjectured that it is sufficient for the base graph to be 3-connected, cubic and Hamiltonian, and also proposed that it is sufficient for the base graph to have a 1-factorisation. They also proved that any potential sufficient conditions are not necessary by showing that the prism of the Petersen graph $P\square K_2$ has a Hamilton decomposition. The Petersen graph does not have a Hamilton cycle or a 1-factorisation, but it does have a Hamilton path. We have already shown that it is not necessary that the base graph has a Hamilton decomposition, and it remains unclear whether the correct generalisation of Alspach and Rosenfeld's Conjecture is for the base graph to be Hamilton decomposable or Hamiltonian, especially in the 4-regular base graph case. In either case, an analogous result here is to find a graph (or family of graphs) which is 4-regular and not Hamiltonian, and which is Hamilton decomposable in its infinite prism. An example of such a graph is given in Figure 3.14. Note that the given graph certainly does not have a Hamilton cycle as there is a cut-vertex c. Note however that the graph does still have a Hamilton path. Similar constructions have been found for other graphs which are semi-Hamiltonian and have a cut-vertex. Perhaps the infinite prism of one of these *generalised butterfly* graphs is always Hamilton decomposable. Given that as yet no graphs which are even-regular, connected, and have the right number of vertices have been shown to *not* have a translation-equivalent Hamilton decomposition of period 3 in their infinite prism, perhaps it is an interesting problem to find an example of such a graph. Perhaps a 4-regular base graph which does not contain a Hamilton path is a good place to start looking. The author has not yet had any success.



(a) The butterfly-like graph B with its Hamilton path in grey.

(b) The canonical path for a Hamilton decomposition of $B\square P_\infty$.

Figure 3.14

Chapter 4

6-regular, two-ended Cayley graphs in general

Theorem 6 has proven that a 6-regular, two-ended Cayley graph with exactly one non-torsion generating element has a Hamilton decomposition as long as it passes the admissibility criteria. We now turn our attention to the general case of a 6-valent, two-ended, connected Cayley graph of infinite order which satisfies the admissibility criteria. Let $S^+ = \{g_1, g_2, g_3\}$ be the set of generating elements of Γ such that $S = S^+ \cup -S^+$ and $S^+ \cap -S^+ = \emptyset$. Assume that $0 \notin S$. Considering the order of the elements of S^+, there are four cases:

1. Two torsion elements, one non-torsion element: Theorem 6 shows that this case always has a Hamilton decomposition.

2. All three elements are torsion: In this case the graph is a disjoint union of finite graphs, so trivially no Hamilton decomposition exists.

3. Zero torsion elements, three are non-torsion.

4. One torsion element, two non-torsion.

In the remainder of this section, we consider the unsolved cases 3 and 4.

4.1 A backtracking algorithm for finding solutions

Backtracking is a general heuristic algorithm for enumerating a set of solutions to a combinatorial problem subject to some constraints [8]. The general strategy is rather intuitive: Begin constructing a potential solution, noting each time a decision is made in the design process. If a decision leads to an incorrect solution, backtrack and make a different decision. If a solution is found, either save it and backtrack, or return that solution and stop. As with almost any problem that requires keeping track of decision points, this process is illuminated with a tree structure. Let the nodes in a rooted tree be partial solutions, with the root an *empty* partial solution. Then the *children* of a node c are any partial solutions which are an extension of c by exactly one step. A node is a *leaf* if it has no children; it is either a solution, or a candidate which has no extensions that would satisfy the constraints of the problem. The efficiency of a backtracking algorithm is determined primarily by three considerations:

1. the number of extensions of each node,

2. the depth of the tree, and

3. the ability to reject partial candidates as close to the root as possible, called *pruning*.

The first two considerations essentially determine the size of a brute-force approach, and the third determines the proportion of brute-force candidates that are actually enumerated by the procedure.

The general procedure for the backtracking algorithm designed to find solutions on d-finite reductions can be summarised with the pseudocode in Algorithm 1, where P is the data specifying the exact d-finite reduction, and c and s are partial solution paths in the d-finite reduction.[1]

Algorithm 1 Recursive Backtracking Algorithm

```
procedure Backtrack(c)
    if reject(P, c) then return        ▷ Prunes a rejected candidate, steps back up to the previous level.
    end if
    if accept(P, c) then output(P, c)  ▷ Recognises a solution and handles it according to output(P, c).
    end if
    s ← first(P, c)                    ▷ Generates the first extension of c.
    while s ≠ NULL do                  ▷ Ends the loop if there are no more children of c.
        bt(s)                          ▷ Recursively runs the backtracking procedure on the child s.
        s ← next(P, s)                 ▷ Generates the next child of c.
    end while
end procedure
```

This backtracking algorithm can then be called on the root candidate (*bt(root(P))*) to find a solution. This procedure relies on six helper algorithms and two objects, which are explained below.

The data P

The object P stores all of the data required to find a solution for a particular generating set and $3h$-finite reduction. The properties of P are as follows:

- w: The size of the finite order subgroup $\mathbb{Z}_w$. Note that it is assumed here that the finite order subgroup is cyclic.
- h: The multiple of 3 specifying $3h$, the height of the finite reduction.
- G: The $3h$-finite reduction.
- S^+: The set of positive generating elements that determines the $3h$-finite reduction.
- e_1: The first edge, the one selected without loss of generality to be in every solution.
- m: A parameter storing the maximum net vertical displacement amongst the generating elements in S^+. This is used to prune candidates which have a current net vertical displacement too large to ever return to the required final vertical displacement of $\pm 3h$.
- D: A dictionary storing the vertical shift for each generating element.

The candidate c

The object c stores data specifying a particular candidate solution. The properties of c are as follows:

- Q: The partial graph in the $3h$-finite reduction.
- R: A subgraph of G storing all the edges that have not yet been considered as extensions of Q.
- e_c: The last edge added to Q.
- t: A vertex in G representing the tail of Q.

[1] One should never reference Wikipedia lightly, but given the universality of the backtracking method, and the article's usefulness, it would be unfair not to cite [38] as a handy introduction to the considerations required to write a backtracking algorithm.

- δ: The current vertical translation of Q.
- t': The previous tail node.
- δ': The vertical translation of the path before e_c was added.

root(P)

root(P) returns the partial candidate at the root of the tree. All other candidates must be an extension of *root(P)* for a complete search. Given the vertex-transitivity of the d-finite reduction, and the requirement that exactly h edges of each edge-type appear in a solution, one edge can be fixed in the root candidate without loss of generality.

reject(P, c)

reject(P, c) should return *true* only if candidate c will not lead to a solution under any extensions. Otherwise it should return *false*. *reject(P, c)* returns *true* under the following circumstances:

- Too many edges in Q of a particular edge-type.
- Q forms a smaller cycle.
- For an edge $e \in E(Q)$, $e + h$ or $e + 2h$ is in $E(Q)$.
- The current vertical translation of Q is too far away from $\pm 3h$ to return to $\pm 3h$ within the $(3hw - E(Q))$ edges left to be added to Q.

Otherwise, it returns *false*.

accept(P, c)

accept(P, c) should return *true* only if c is a solution, and return *false* otherwise. *accept(P, c)* returns *true* when a candidate satisfies all the criteria in Claim 4, and *false* otherwise.

first(P, c)

first(P, c) generates the first extension from candidate c. The below pseudocode summarises the process.

Algorithm 2 The first extension of c

procedure FIRST(P, c)
  if $|E(Q)| = 3hw$ **then** return None ▷ If the candidate has the maximum number of edges, then it has no extensions.
  end if
  $E \leftarrow \{tv \mid tv \in G, tv \notin Q, deg(v) < 2\}$ ▷ E is the set of edges incident with the tail t such that the edge is not in Q and v is not already visited by Q.
  $R \leftarrow G[E]$ ▷ The *remaining edges* R is the subgraph of G induced by E.
  if $|E(R)| = 0$ **then** return None ▷ If R has no edges, there are no possible extensions.
  end if
  $e_c \leftarrow$ uniform random selection from $E(R)$
  $E(R) \leftarrow E(R) \setminus e_c$ ▷ Remove e_c from R.
  $E(Q) \leftarrow E(Q) \cup e_c$ ▷ Add e_c to Q.
  $t' \leftarrow t$ ▷ Update previous tail node.
  $t \leftarrow v \mid e_c = tv$ ▷ Update tail node.
  $\delta' \leftarrow \delta$ ▷ Update vertical translation at new t'.
  $\delta \leftarrow \delta + \delta(e_c)$ ▷ Update vertical translation at new t
  return the new candidate object
end procedure

***next*(P, c)**

next(P, c) generates the next extension after candidate c. The below pseudocode summarises the process.

Algorithm 3 The next extension of c

procedure $\textsc{Next}(P, c)$
  $E(Q) \leftarrow E(Q) \setminus e_c$ ▷ Remove the old extension.
  if $|E(R)| = 0$ **then** return None ▷ Return None if there are no more possible extensions.
  end if
  $e_c \leftarrow$ uniform random selection from $E(R)$ ▷ Pick a new random edge from the remaining extensions.
  $E(R) \leftarrow E(R) \setminus e_c$ ▷ Remove the new extension from R.
  $E(Q) \leftarrow E(Q) \cup e_c$ ▷ Add the new extension to Q.
  $t \leftarrow v \mid e_c = t'v$ ▷ Update tail node.
  $\delta \leftarrow \delta' + \delta(e_c)$ ▷ Update vertical translation at new t.
  return the new candidate object
end procedure

***output*(P, c)**

output(P, c) should handle a solution c as required. By default *output(P, c)* just halts the program and prints the solution, given that most solutions are isomorphic due to the symmetry of the graph.

4.2 A complete search on $\mathbb{Z}_3 \oplus \mathbb{Z}$

4.2.1 Isomorphisms

If we limit ourselves to $w = 3$ and a period of 3, then we can reasonably concisely determine all non-isomorphic generating sets, and run the search algorithm on these generating sets to comprehensively determine all solutions where $w = 3$, $k = 3$, and the period is 3. In order to do this, we must first build a notion of what makes a pair of generating sets isomorphic.

Definition 13 (Isomorphic positive generating sets)**.** A generating set S will only be identified by S^+, noting that $S^+ \cup -S^+ = S$. We will consider two positive generating sets $S_1^+, S_2^+ \subset \Gamma \setminus \{0\}$ isomorphic in the natural way, where $S_1^+ \simeq S_2^+ \iff Cay(\Gamma, S_1) \simeq Cay(\Gamma, S_2)$.

We now identify two useful isomorphisms between positive generating sets, and a morphism that preserves the properties of canonical path under certain conditions.

Sign

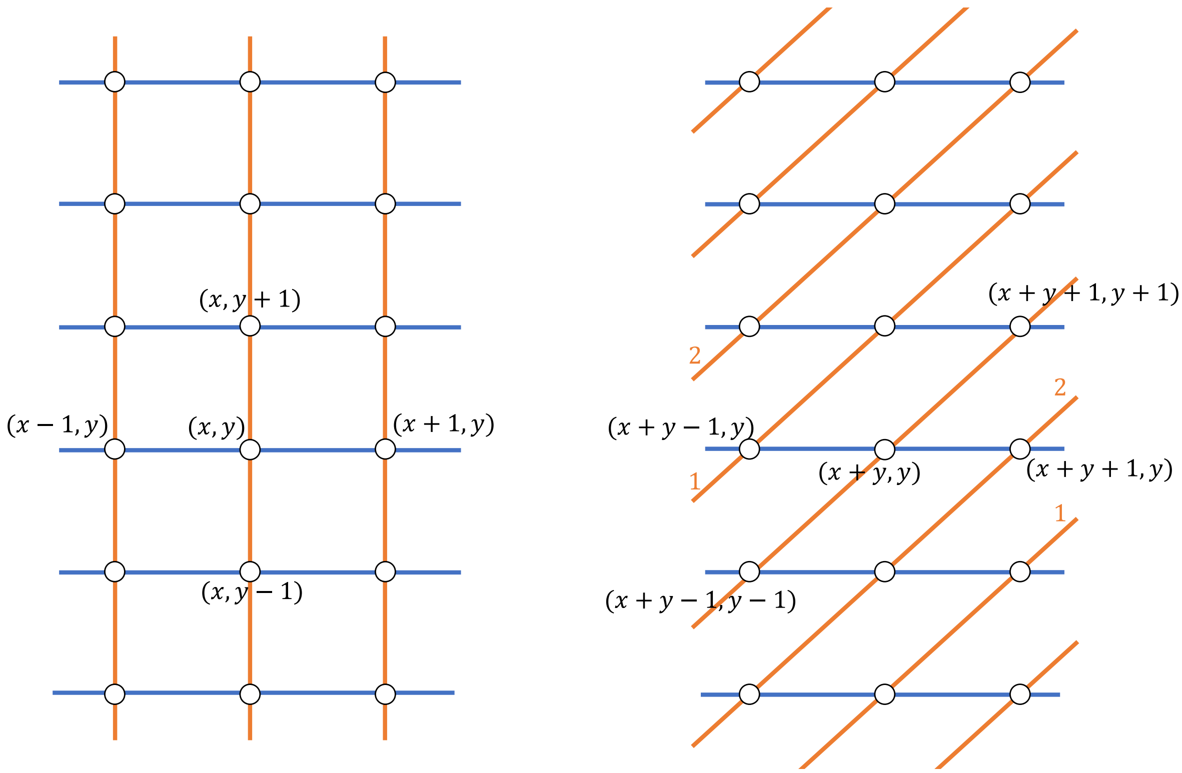
Two positive generating sets S_1^+ and S_2^+ obviously generate isomorphic Cayley graphs if $S_1^+ \cup -S_1^+ = S_2^+ \cup -S_2^+$. Note that this implies that for any element $s \in S_1^+$, $-s$ or s is in S_2^+.

A morphism between generating sets for a given finite reduction height

Let $S_1^+ = \{g_1, g_2, g_3\}$ be a generating set with a solution on a $3h$-finite reduction where the direction of every g_1-edge and g_2-edge in the canonical path is the same. Then $S_2^+ = \{g_1 + (0, 3h), g_2 - (0, 3h), g_3\}$ also has a solution, as the two generating sets will have isomorphic $3h$-finite reductions, so the canonical cycle will be valid for both. Further, the net vertical displacement will still be $\pm 3h$ as the w edges of type $g_1 + (0, 3h)$ will increase the net vertical displacement by $3hw$, but this will be cancelled out by the w edges of type $g_2 - (0, 3h)$. The generating sets identified below which have solutions with this property are S_A^+, S_C^+, and S_F^+. The solutions for these generating sets can be used to determine solutions for another generating set which can be attained by applying the above morphism.

Twist

A less obvious isomorphism between generating sets is given by a twisting notion. For an example, consider the two positive generating sets $S_1^+ = \{(0,1),(1,0)\}$ and $S_2^+ = \{(1,1),(1,0)\}$ and group $\mathbb{Z}_3 \oplus \mathbb{Z}$. (See Figure 4.1) Although $(0,1)$ and $(1,1)$ are not equivalent, $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, S_1)$ and $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, S_2)$ are isomorphic through the following function: for all $(x,y) \in V(G_\infty)$, $f((x,y)) = (x+y,y)$. For an arbitrary vertex (x,y), its neighbours in $Cay(\Gamma, S_1)$ are $(x,y+1),(x,y-1),(x+1,y),(x-1,y)$. To show that f is an isomorphism, note that f has the following actions: $f((x,y)) = (x+y,y), f((x,y+1)) = (x+y+1,y+1), f((x,y-1)) = (x+y-1,y-1), f((x+1,y)) = (x+y+1,y), f((x-1,y)) = (x-1+y,y)$, which are the neighbours of $(x+y,y)$ in $Cay(G_\infty, S_2)$. Thus, the two Cayley graphs are isomorphic, and so S_1^+ and S_2^+ are isomorphic.



(a) $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,0)\})$. (b) $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(1,1),(1,0)\})$.

Figure 4.1: An illustration of two graphs isomorphic through twist.

More generally, the isomorphism $f((x,y)) = (x+ay,y)$ for some $a \in \mathbb{Z}$ can be called an isomorphism with twist a. However, with $w = 3$, The only non-trivial twists are $a = 1$ and $a = -1 = 2$.

4.2.2 Non-isomorphic generating sets on a 3-finite reduction in $\mathbb{Z}_3 \oplus \mathbb{Z}$

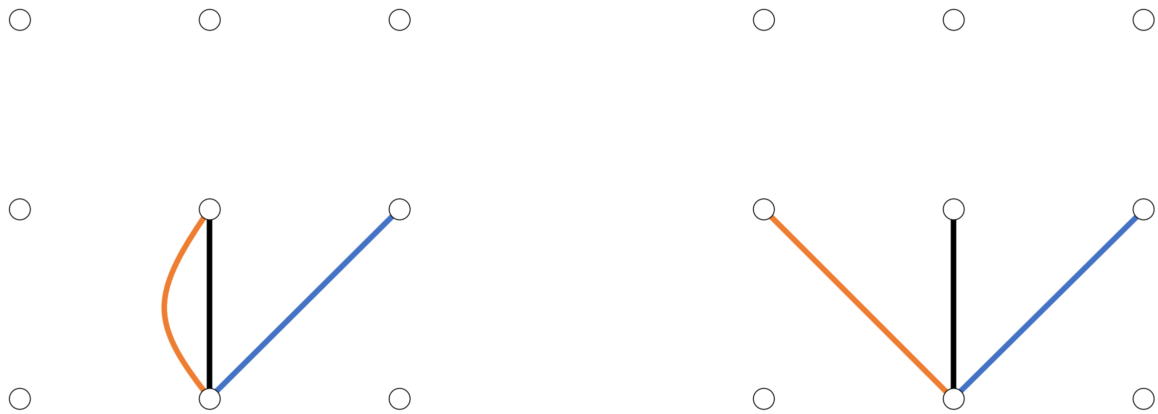
In order to categorise all non-isomorphic generating sets of a 3-finite reduction in $\mathbb{Z}_3 \oplus \mathbb{Z}$, we first break into the two cases based on the number of torsion generating elements. First consider the case where we have no torsion generating elements and three non-torsion generating elements.

No torsion generating elements

In order to be connected, there must be a generating element of the form $(x,1)$ for some $x \in \mathbb{Z}_3$. In a general $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ graph two non-torsion elements i.e. $\{(0,2),(1,3)\}$ can generate $\Gamma_{\text{fin}} \oplus \mathbb{Z}$, but in a 3-finite reduction the non-torsion component of each element is reduced modulo 3 to either 0, 1, or 2. A generating element of the form $(x,0)$ corresponds to a non-torsion component, and $(x',2)$ cannot generate the group without an element of the form $(x',1)$ either. Due to the twisting isomorphism we can assume without loss of generality that this generating element is $(0, 1)$. Note that at least one of the remaining generating elements then requires a torsion component, otherwise the graph is not connected. As the remaining generating elements are also non-torsion, they are of the form $(x, 1)$ or $(x, 2)$. Note that in the case where we take the remaining elements to be $(x, 1)$ and $(x', 2)$, the non-torsion component-sum will be even, and thus the Cayley graph will not satisfy the admissibility criteria, so we exclude this case.

First consider the case where every generating element is of the form $(x, 1)$. The two non-isomorphic generating sets are represented in Figure 4.2, and below:

$$S_A^+ = \{(0, 1), (0, 1), (1, 1)\}$$
$$S_B^+ = \{(0, 1), (1, 1), (-1, 1)\}$$



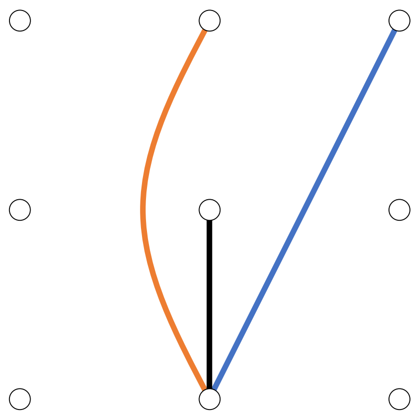
(a) $S_A^+ = \{(0, 1), (0, 1), (1, 1)\}$. (b) $S_B^+ = \{(0, 1), (1, 1), (-1, 1)\}$.

Figure 4.2: Non-isomorphic generating sets where every element is of the form $(x, 1)$.

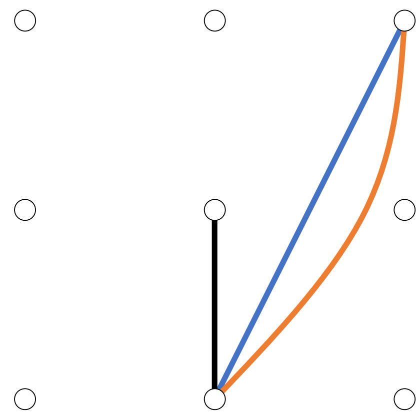
As $(0, 1)$ is fixed and tripling $(0, 1)$ would lead to a disconnected graph, the two sets correspond to the choice of either doubling one of the generating elements in the case of S_A^+ or having three unique elements in the case of S_B^+.

Alternatively, consider the case where the other two generating elements are of the form $(x, 2)$, alongside the necessary element $(0, 1)$. The three non-isomorphic generating sets are represented in Figure 4.3 and below:

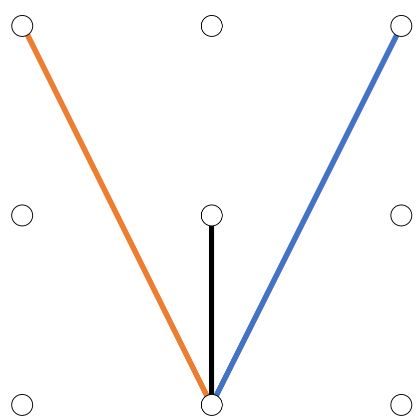
$$S_C^+ = \{(0,1), (0,2), (1,2)\}$$
$$S_D^+ = \{(0,1), (1,2), (1,2)\}$$
$$S_E^+ = \{(0,1), (1,2), (-1,2)\}$$



(a) $S_C^+ = \{(0,1), (0,2), (1,2)\}$.



(b) $S_D^+ = \{(0,1), (1,2), (1,2)\}$.



(c) $S_E^+ = \{(0,1), (1,2), (-1,2)\}$.

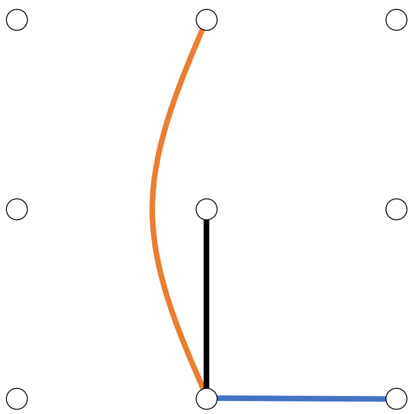
Figure 4.3: Non-isomorphic generating sets where two elements are of the form $(x, 2)$.

As $(0, 1)$ is fixed, if the remaining two elements were both $(0, 2)$ then the generating elements would all be aligned, and the graph would again be disconnected. The three sets then correspond to the choice of having one element aligned with $(0, 1)$ and one element not aligned as in S_C^+, and if neither of the remaining elements is aligned with $(0, 1)$, whether the remaining two should be aligned with each other (as in S_D^+) or not (as in S_E^+).

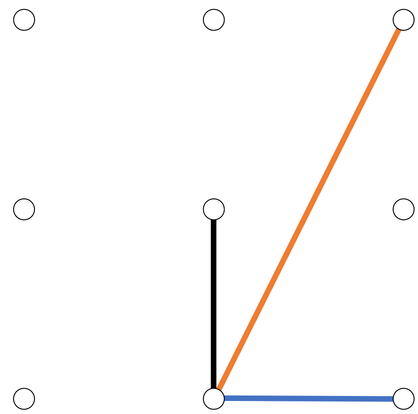
One torsion generating element

Finally, consider the case of one torsion generating element. Without loss of generality this element can be $(1, 0)$. As the generating set must satisfy the admissibility criteria, the remaining element must be of the form $(x, 2)$, otherwise the non-torsion component-sum would be even. The three non-isomorphic generating sets are represented in Figure 4.4 and below:

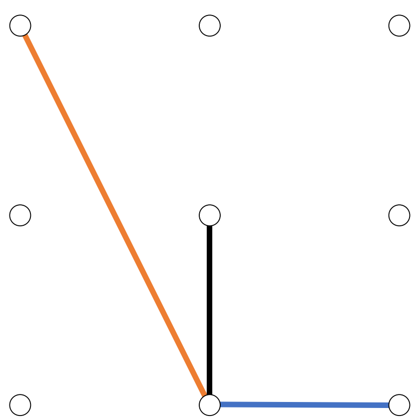
$$S_F^+ = \{(0,1),(0,2),(1,0)\}$$
$$S_G^+ = \{(0,1),(1,2),(1,0)\}$$
$$S_H^+ = \{(0,1),(-1,2),(1,0)\}$$



(a) $S_F^+ = \{(0,1),(0,2),(1,0)\}$.



(b) $S_G^+ = \{(0,1),(1,2),(1,0)\}$.



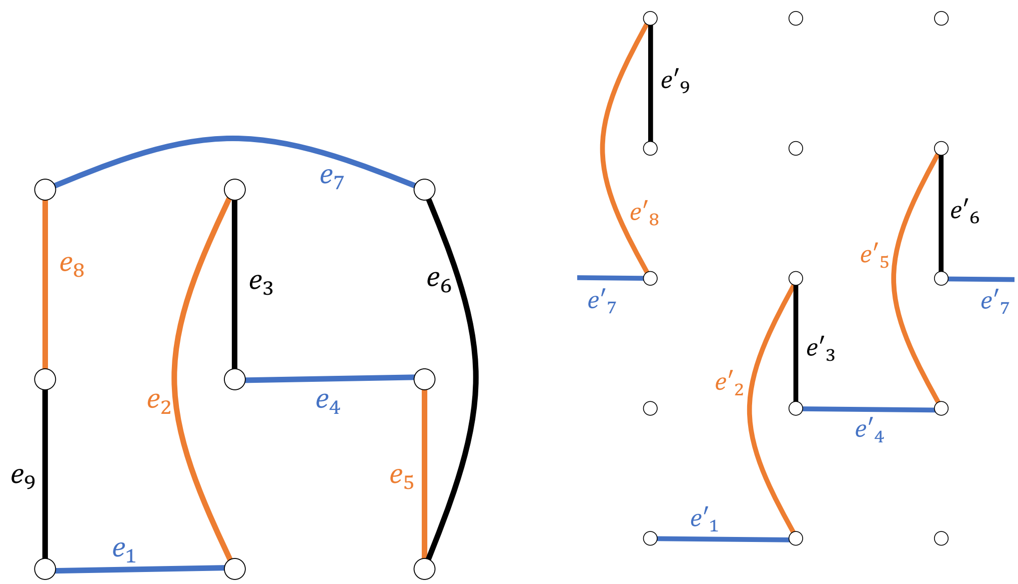
(c) $S_H^+ = \{(0,1),(-1,2),(1,0)\}$.

Figure 4.4: Non-isomorphic generating sets where one element is torsion.

The three generating sets correspond to the choice of whether the third generating element $(x, 2)$ should align with $(0, 1)$, $(1, 0)$ or neither.

4.2.3 Results

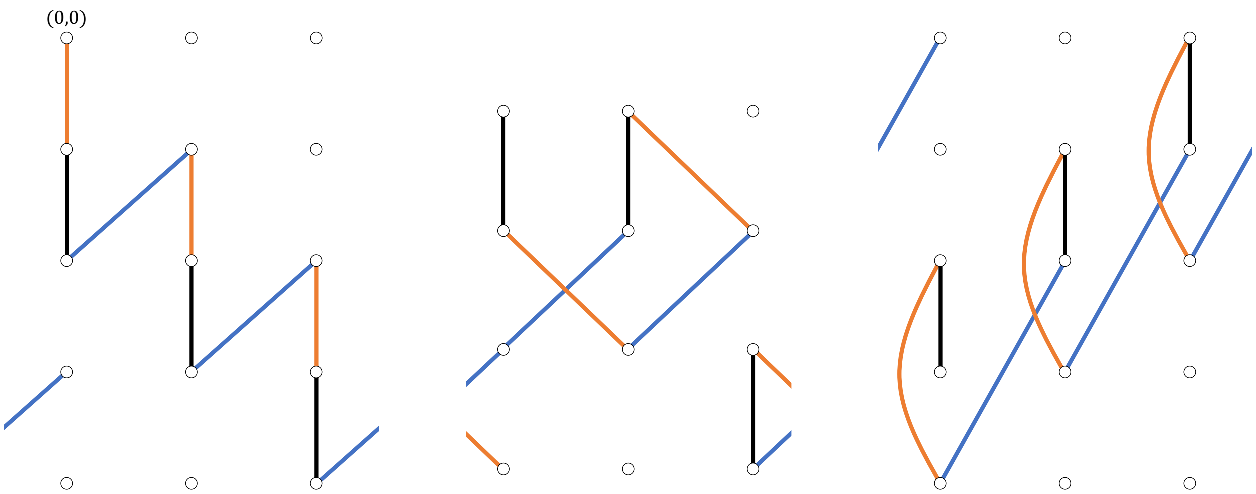
We now provide a canonical path for each of the generating sets in Section 4.2.2. Note that the algorithm finds a solution in a finite reduction, but we can interpret the solution in the finite reduction as a canonical path. An example of the finite reduction solution the algorithm finds for $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, \pm S_F^+)$ is provided in Figure 4.5a and the canonical path solution is provided in Figure 4.5b. The edges have been labelled to indicate which edges are equivalent to each other, where e_i in the finite reduction and e'_i in $Cay(\Gamma_{\text{fin}} \oplus \mathbb{Z}, S)$ are equivalent. We will continue to represent the edge-types with the colour established in Section 4.2.2 as an easy visual check that the right number of edges of each edge-type have been used. We present each canonical path without written proof that it is a canonical path; the visual representation stands as a proof in and of itself that each path is a canonical path that satisfies the sufficient criteria established in Claim 4 to generate a Hamilton decomposition into double-rays. The canonical paths are presented in Figure 4.6 and Figure 4.7. Note in the case of S_D^+ that a solution does not exist with a period of 3 or 6, however one does exist with a period of 9. Why exactly this is the case is not really clear. It also stands as a warning against assuming that if the backtracking algorithm does not find a solution for small periods, then no periodic solution exists at all, even for large periods.



(a) The solution in the finite reduction of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_F^+)$.

(b) The finite reduction solution extended to a canonical path in $Cay(\mathbb{Z}_3, \oplus\mathbb{Z}, \pm S_F^+)$.

Figure 4.5

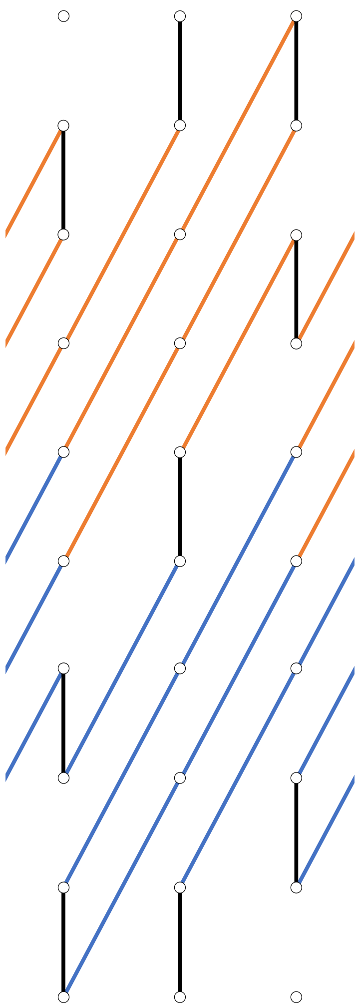


(a) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_A^+)$.

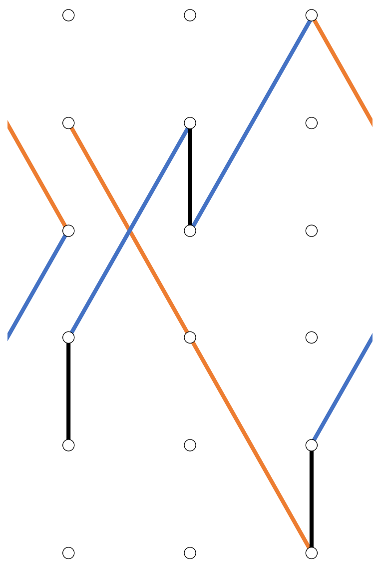
(b) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_B^+)$.

(c) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_C^+)$.

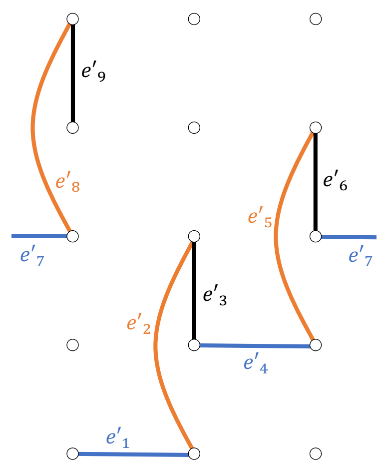
Figure 4.6



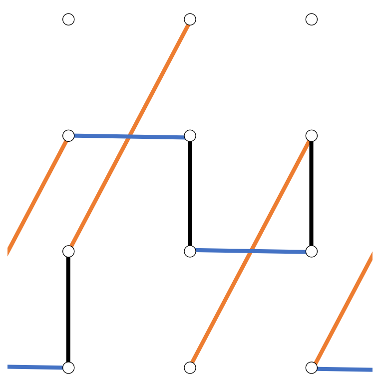
(a) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_D^+)$.



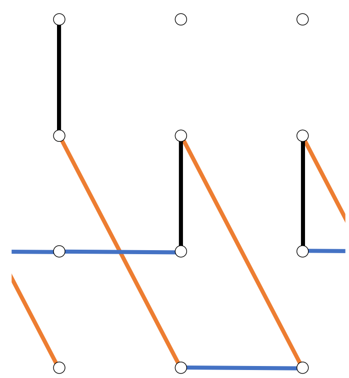
(b) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_E^+)$.



(c) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_F^+)$.



(d) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_G^+)$.



(e) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_H^+)$.

Figure 4.7

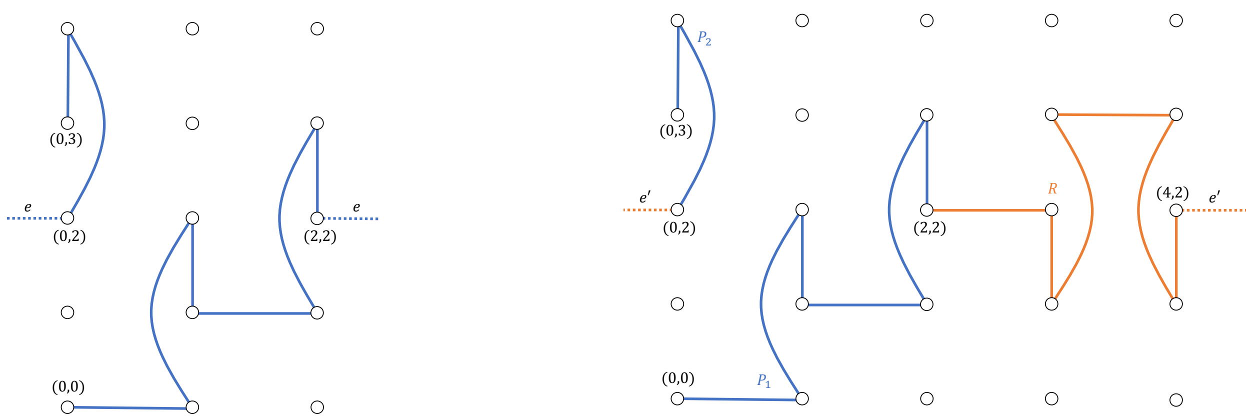
4.3 Generating sets with an infinite family of solutions on $\mathbb{Z}_{2i+1} \oplus \mathbb{Z}$

Claim 7. *$Cay(\mathbb{Z}_{2i+1} \oplus \mathbb{Z}, \pm\{(1,0),(0,1),(0,2)\})$ is Hamilton decomposable for all $i \geq 1$.*

Proof. The proof is by induction. First, note that the base case where $i = 1$ is already proven above in Section 4.2.3. In particular, it is generating set $\pm S_F^+$, and the canonical path is shown in Figure 4.7c. Next, assume that $G = Cay(\mathbb{Z}_{2j+1} \oplus \mathbb{Z}, \pm\{(1,0),(0,1),(0,2)\})$ is Hamilton decomposable and prove that $G' = Cay(\mathbb{Z}_{2j+3} \oplus \mathbb{Z}, \pm\{(1,0),(0,1),(0,2)\})$ is Hamilton decomposable. Let Q be the canonical path from the decomposition of G, without loss of generality starting at $(0,0)$ and ending at $(0,3)$. As we are building up from a base case in a constructive way, we can assume that Q is of the form implied by our construction. (See Figure 4.8.) If $M = \{\{(2j,a),(0,a)\} \mid a \in \mathbb{Z}, \{(2j,a),(0,a)\} \in E(G)\}$, and $W = \{(x,y) \in V(G') \mid x \in \{0,1,2,\ldots,2j\}\}$, then $G \setminus M$ is isomorphic to $G'[W]$ under the identity map $f : V(G \setminus M) \mapsto V(G'[W]) : f((x,y)) = (x,y)$. Let $Q' = f(Q)$ be the image of Q in G'. As Q is a canonical path in G, it only contains one edge e from M, so Q' is a pair of paths, with one path P_1 starting at $(0,0)$ and ending at $(2j,a)$ and the other path P_2 starting at $(0,a)$ ending at $(0,3)$. That P_1 ends at $(2j,a)$ and P_2 starts at $(0,a)$ is guaranteed by the fact that Q was built by this construction starting with the path in Figure 4.7c. As Q was a canonical path in G of period 3, P_1 and P_2 use one of each edge type for all edges except those between the $2j$ and 0 columns of vertices. It then suffices to build a path R between $(2j,a)$ and $(0,a)$, which uses one of each remaining edge type. Let R be as follows: (See Figure 4.8b)

$$R = [(2j,a),(2j+1,a),(2j+1,a-1),(2j+1,a+1),(2j+2,a+1),(2j+2,a-1),(2j+2,a),(0,a)]$$

.



(a) $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,0)\})$.

(b) $Cay(\mathbb{Z}_5 \oplus \mathbb{Z}, \pm\{(1,1),(1,0)\})$.

Figure 4.8: The first extension in the induction in the proof of Claim 7.

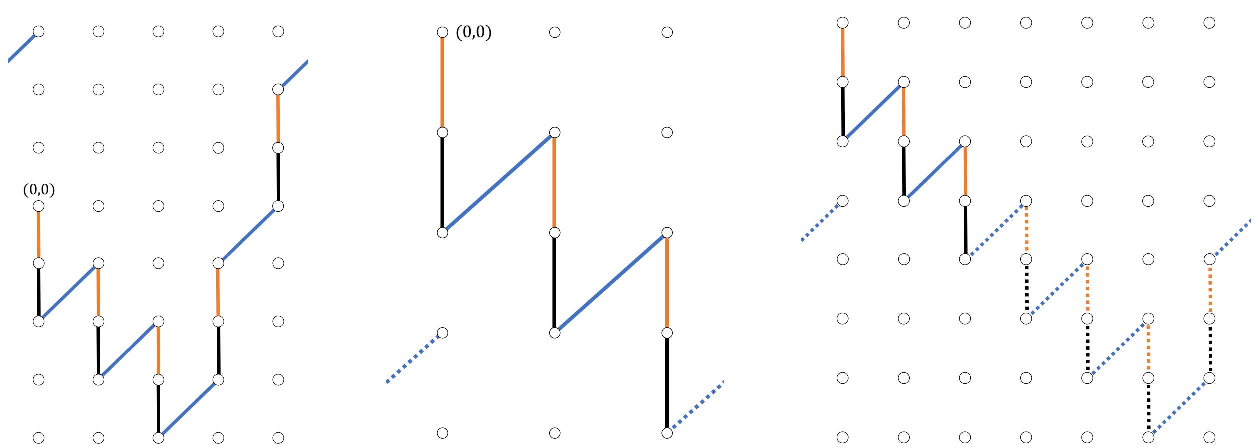
$[P_1, R, P_2]$ is then a canonical path for G', so G' is Hamilton decomposable. □

Claim 8. *$Cay(\mathbb{Z}_{2i+1} \oplus \mathbb{Z}, \pm\{(0,1),(0,1),(1,1)\})$ is Hamilton decomposable for all $i \geq 1$.*

Proof. The proof mirrors that of the proof of Claim 7, with the variation being that the inductive path R in this case will increase the width by four, not two. Thus, two base cases are required, the first ($i = 1$) is shown in Figure 4.6a, and the second ($i = 2$) is shown in Figure 4.9a. Note that, as presented, if we consider both paths as starting at the marked vertex $(0,0)$ and beginning with the same pattern, then in the $i = 1$ case the path has a net vertical displacement of -3, but in the $i = 2$ case the path has a net vertical displacement of $+3$. As the induction step increments i by two, we will treat these as two separate cases when necessary.

Now assume that $G = Cay(\mathbb{Z}_{2j+1} \oplus \mathbb{Z}, \pm\{(1,0),(1,0),(1,1)\})$ is Hamilton decomposable and prove that $G' = Cay(\mathbb{Z}_{2j+5} \oplus \mathbb{Z}, \pm\{(1,0),(1,0),(1,1)\})$ is Hamilton decomposable. Let Q be the canonical path from the decomposition of G. As we are again building up from a base case in a constructive way, we can assume that Q is of the form implied by our construction. (See Figure 4.9a and Figure 4.9b.) If $M = \{\{(2j,a),(0,a+1)\} \mid a \in \mathbb{Z}, \{(2j,a),(0,a+1)\} \in E(G)\}$, and $W = \{(x,y) \in V(G') \mid x \in \{0,1,2,\ldots,2j\}\}$, then as in the previous proof $G \setminus M$ is isomorphic to $G'[W]$ under the identity map $f : V(G \setminus M) \mapsto V(G'[W]) : f((x,y)) = (x,y)$. Let $Q' = f(Q)$ be the image of Q in G'. As Q is a canonical path in G of period 3, it only contains one edge e from M. Without loss of generality, assume that e is of the form $\{(2j,2),(0,3)\}$ if i is even, and $\{(2j,-4),(2j,-3)\}$ if i is odd. In this case, as e is the last edge in Q in our construction, Q' is one path which now starts at $(0,0)$ and ends at $(2j,2)$ or $(2j,-4)$ depending on the parity of i. In either case, as Q was a canonical path in G of period 3, Q' uses one of each edge type for all edges except those between the $2j$ and 0 columns of vertices. It then suffices to build a path R between $(2j,a)$ and $(0,a+1)$ which uses one of each remaining edge type. Let R be as follows: (See Figure 4.9c)

$$\begin{aligned}R = [&(2j,a),(2j+1,a+1),(2j+1,a),(2j+1,a-1),(2j+2,a),(2j+2,a-1),(2j+2,a-2),\\&(2j+3,a-1),(2j+3,a-2),(2j+3,a-3),(2j+4,a-2),(2j+4,a-1),(2j+4,a),(0,a+1)]\end{aligned}$$



(a) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_5 \oplus \mathbb{Z}, \pm S_A^+)$.

(b) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm S_A^+)$.

(c) The canonical path for a Hamilton decomposition of $Cay(\mathbb{Z}_7 \oplus \mathbb{Z}, \pm S_A^+)$, with R in dotted lines.

Figure 4.9: The base cases and the first extension in the induction in the proof of Claim 7.

$[Q', R]$ is then a canonical path for G', so G' is Hamilton decomposable. $\square$

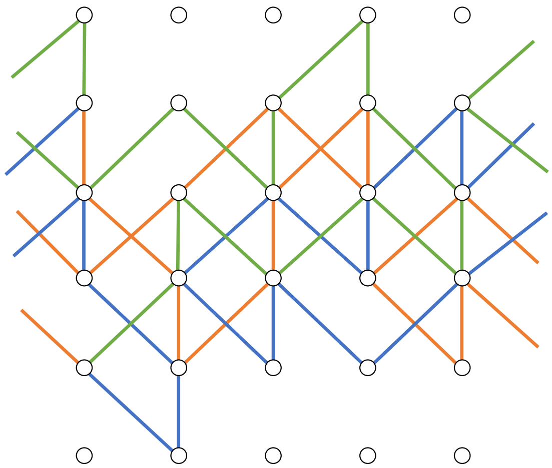


Figure 4.10: One period of the three double-rays that Hamilton decompose $Cay(\mathbb{Z}_5 \oplus \mathbb{Z}, \pm\{(0,1),(1,1),(1,-1)\}$. Note that they are not equivalent under translation.

It appears that S_A^+ and S_F^+ are the only two generating sets from the enumeration of non-isomorphic generating sets in Section 4.2.2 which have an easy, constructive canonical path with period 3 for all $Cay(\mathbb{Z}_{2i+1} \oplus \mathbb{Z}, S)$. For the other six generating sets identified in Section 4.2.2, the algorithm has been run to verify that no solution of period 3 exists for $Cay(\mathbb{Z}_5 \oplus \mathbb{Z}, S)$, and the algorithm runs to completion in each case rather quickly. When the generating sets are checked for period 3 solutions on $Cay(\mathbb{Z}_7 \oplus \mathbb{Z}, S)$, no solution is found either, but it takes on average 23 minutes on Google Colab to run each complete search. For period 3 solutions on $Cay(\mathbb{Z}_9 \oplus \mathbb{Z}, S)$, it takes longer than can be reliably run to completion for every generating set with the given resources, and the stability of those resources. For that reason, it is more practical to investigate one example in-depth to try and find solutions, than to do a complete search on every example, and that is what is done in the following section.

4.4 $Cay(\mathbb{Z}_w \oplus \mathbb{Z}, \pm\{(0,1),(1,1),(-1,1)\}$: A pariah

$Cay(\mathbb{Z}_3 \oplus \mathbb{Z}, \pm\{(0,1),(1,1),(-1,1)\})$ has a Hamilton decomposition, as was demonstrated by the canonical path in Figure 4.6b in Section 4.2.3. However, a complete backtracking search fails to find a periodic translation on $Cay(\mathbb{Z}_w \oplus \mathbb{Z}, \pm\{(0,1),(1,1),(-1,1)\})$ for $w = 5$ or $w = 7$ and a period of 3. Higher widths fail to find a solution, but the algorithm exceeds reasonable computation time and has not been run to completion. This still suggests that no solution exists in these cases, as when a solution does exist in other examples, one is normally found significantly faster than a complete search, due to the symmetries of the finite reduction. However, if the condition that all paths are equivalent under translation is dropped, a solution with three double-rays of period 3 can be found, as is demonstrated in Figure 4.10. (Note that this was not found by computer. One of the three double-rays was found by the algorithm modified to not check for intersecting edges under translation, and the remaining two paths were found by hand by the author.) When one searches for periodic translation solutions on $w = 5$ and a period of 6, a solution is found, as is demonstrated in Figure 4.12. (This was found by computer.)

Thus, $Cay(\mathbb{Z}_5 \oplus \mathbb{Z}, \pm\{(0,1),(1,1),(-1,1)\})$ stands as a good counterexample to a potential hypothesis that it is sufficient only to look for translation solutions of period 3. It also demonstrates that the periodic assumption does not imply that the double-rays will all be equivalent under translation. It remains open, however, as to whether the translation assumption implies periodicity, and whether or not there is any connection between the period 3, $w = 3$

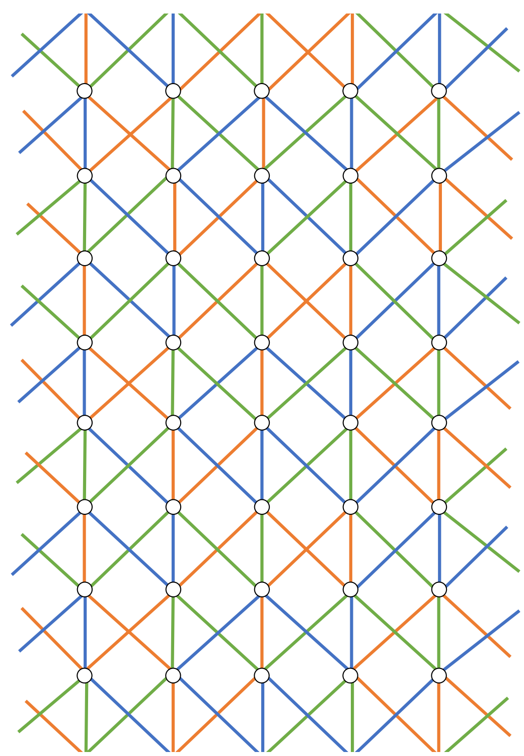


Figure 4.11: The paths in Figure 4.10 extended to double-rays which form a Hamilton decomposition.

solution and the period 6, $w = 5$ solution. No solution of period 6 for $w = 7$ can be found, and the algorithm is simply not efficient enough to evaluate whether or not there is a solution of period 9 for $w = 7$. The algorithm has been run for a total of over ten hours in this case, with no solution found. However under the same period and width conditions and similar run-time the algorithm also fails to find a solution for generating sets S_A^+ and S_F^+ which are known to have a solution. This demonstrates the practical limits of the algorithm with current efficiency and hardware available, so trying higher periods would be even less instructive.

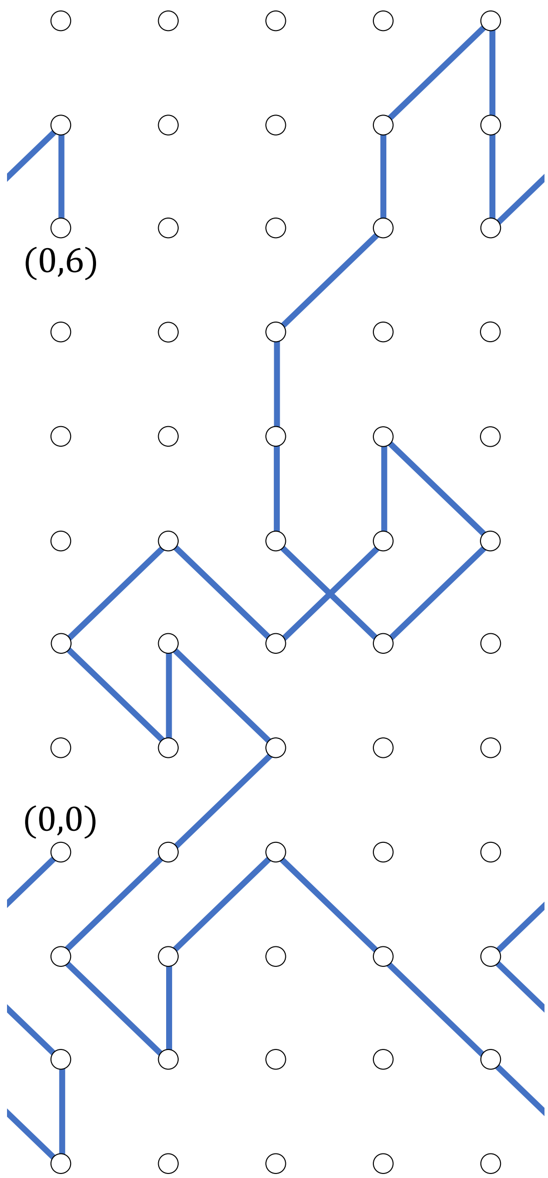


Figure 4.12: The canonical path for a translation-equivalent Hamilton decomposition of $Cay(\mathbb{Z}_5 \oplus \mathbb{Z}, \pm S_B^+)$ with period 6.

Chapter 5

Conclusion

In Chapter 1, we introduced three folklore conjectures on Hamilton decompositions of finite graphs, and considered how to generalise these to infinite graphs. The notion of the Hamilton decomposition of an infinite graph was developed, and a topological obstruction was identified which prevents two-ended infinite graphs from having a Hamilton decomposition. This obstruction was classified completely for two-ended Cayley graphs with Lemma 1, and for $G \Box P_\infty$ graphs with Corollary 2. Claim 3 identified a connection between two-ended Cayley graphs with one non-torsion generating element and $G \Box P_\infty$ graphs, and a set of sufficient criteria for a Hamilton decomposition into periodic translation-equivalent double-rays was developed in Section 1.8.

It was proven in Chapter 2 that if G has a Hamilton decomposition into one, two or three cycles and passes the admissibility criteria, then $G \Box P_\infty$ has a Hamilton decomposition into double-rays. In the case of G being decomposed into three Hamilton cycles, the proof assumed that G was simple, and this assumption has not yet been remedied, but it should be avoidable. One of the main results of the thesis is Theorem 6, which showed that every 6-regular, two-ended Cayley graph with exactly one non-torsion generating element has a Hamilton decomposition into double-rays, as long as it passes the admissibility criteria. This result could also be extended to 8-regular Cayley graphs, but it is conditional on Alspach's conjecture for finite graphs being true for 6-regular graphs, and relies on the equivalent 6-regular base graph being simple.

In Chapter 3, it was shown that it is not necessary for a 4-regular graph G to have a Hamilton decomposition for $G \Box P_\infty$ to have a Hamilton decomposition. An infinite family of Hamiltonian graphs with a 2-edge cut were shown to have a Hamilton decomposition in their infinite prism, and an example was given of a semi-Hamiltonian graph with a Hamilton decomposition in its infinite prism. No example has been found of a simple 4-regular graph G such that $G \Box P_\infty$ does not have a translation-equivalent Hamilton decomposition of period 3. From the results in Chapter 3, it is not clear whether the right assumption for the generalisation of Alspach and Rosenfeld's Conjecture to a 4-regular base graph G, is for G to be Hamilton decomposable, or Hamiltonian. A counterexample would help settle this, but none is currently known.

In Chapter 4, we attempted to classify the remaining cases of 6-regular, two-ended Cayley graphs, that being those with two or three non-torsion generating elements. A backtracking algorithm was developed for finding periodic, translation equivalent Hamilton decompositions, or concluding that none exists for a given period. A set of non-isomorphic generating sets in $\mathbb{Z}_3 \oplus \mathbb{Z}$ was identified, and each of these was shown to have a Hamilton decomposition. In Section 4.3 it was shown that two of these generating sets admit an infinite family of translation-equivalent Hamilton decompositions of period 3 for all $\mathbb{Z}_{2j+1} \oplus \mathbb{Z}$, and the backtracking algorithm was used to show that no such infinite families exist for the other generating sets. One of these badly behaved generating sets was investigated in further detail, and in the Cayley graph of $\mathbb{Z}_5 \oplus \mathbb{Z}$, it was shown to have a non-translation Hamilton decomposition of period 3, and a translation-equivalent Hamilton decomposition of period 6. The algorithm has not been run long enough to determine if similar decompositions exist for $\mathbb{Z}_7 \oplus \mathbb{Z}$.

Appendix A

Code

The below code was written in Google Colab running Python version 3.6.9 [33]. It relies heavily on the NetworkX package [21] and NumPy package [22]. Note that the code below is written with print statements commented out for speed. They have been left in as vestiges for clarity as the print statements help keep track of the progress of the algorithm. Additionally, the order of the coordinates is reversed (i.e. $\Gamma = \mathbb{Z} \oplus \mathbb{Z}_w$) from the rest of the thesis as this code was written before establishing the standard that $\Gamma = \mathbb{Z}_w \oplus \mathbb{Z}$.

A.1 Packages imported

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
import sys
```

A.2 Backtracking helper methods

A.2.1 Recursive backtracking function

```
 # the recursive backtracking algorithm
 def bt(c, sol_list):
 # P: a tuple of static instance-specific data, of the form:
 # P = (w, k, G, gen_set, first_edge, max_shift, shift_dict)
 # int w: the width of the graph
 # int k: the multiple height of the graph
 # MultiGraph G: the graph of interest
 # list gen_set: the list of generating elements and their colour
 # list first_edge: a list of two vertices and a colour
 # int max_shift: the maximum vertical shift possible per edge
 # dict shift_dict: a dictionary storing the vertical shift for each colour
 #
 # c: a tuple of candidate-spcific information, of the form:
 # c = (partial, remaining, candidate_edge, tail_node, vertical_translation,
 # second_to_last, s_to_l_translation)
 # MultiGraph partial: the partial graph under construction
 # MultiGraph remaining: the edges as yet not considered
 # edge candidate_edge: the last edge added to partial
 # node tail_node: the tail node of partial
 # int vertical_translation: the implied vertical translation in the infinite
 # path of the tail_node
 # node second_to_last: the node that was the previous tail_node
 # int s_to_l_translation: the vertical translation of the second_to_last node
 if reject(P, c):
```

```
    return
  if accept(P, c):
    sol_list = output(P, c, sol_list)
  s = first(P, c)
  while not s is None:
    bt(s, sol_list)
    s = next_one(P, s)
```

A.2.2 *root(P)*

```
def root(P):
#print("creating root")
partial = nx.MultiGraph()
first_edge = P[4]
#print("first edge", first_edge)
G = P[2]
partial.add_nodes_from(G)
partial.add_edge(first_edge[0], first_edge[1], key = first_edge[2], colour = first_edge[2])
remaining = G.copy()
candidate_edge = None
tail_node = first_edge[1]
second_to_last = first_edge[0]
vertical_translation = tail_node[0] - second_to_last[0]
s_to_l_translation = 0
#print("vertical_translation", vertical_translation)
#print("partial at root", partial.edges.data())
#print("candidate at root", candidate_edge)
c = (partial, remaining, candidate_edge, tail_node, vertical_translation, second_to_last,
    s_to_l_translation)
return c
```

A.2.3 *reject(P, c)*

```
def reject(P, c):
partial = c[0]
number_of_edges = partial.number_of_edges()
if number_of_edges is 0:
  # empty partial graph cannot be rejected
  return False
else:
  # check for too many edges of one colour
  red_count = 0
  blue_count = 0
  green_count = 0
  w = P[0]
  k = P[1]
  maxim = w*k
  for e in list(partial.edges.data()):
    colour = e[2]['colour']
    if colour is 'blue':
      blue_count += 1
      if blue_count > maxim:
        return True
    elif colour is 'red':
      red_count += 1
      if red_count > maxim:
        return True
      elif colour is 'green':
      green_count += 1
      if green_count > maxim:
```

```
        return True
      else:
        raise Exception("invalid colour")

    # check if a smaller cycle than required is created
  try:
    cycle = list(nx.find_cycle(partial))
  except:
    # no cycle is found
    pass
  else:
    if len(cycle) < 3*maxim:
      # cycle is too small
      return True

  # check for distribution of colours amongst vertical translations
  for e in list(partial.edges.data()):
    # for each edge, check if other vertical translations of it are the
    # same colour
    e_u = e[0]
    e_v = e[1]
    e_col = e[2]['colour']
    for i in range(1,3):
      # check vertical translations by 1 or 2
      new_u = ((e_u[0]+i*k)%(3*k), e_u[1])
      new_v = ((e_v[0]+i*k)%(3*k), e_v[1])
      if partial.has_edge(new_u, new_v, key = e_col):
        # matching translation found
        return True

  # check for vertical_translation too far away to return to +3 by end
  # of cycle
  edges_left = 3*k*maxim - number_of_edges
  max_shift = P[5]
  vertical_translation = c[4]
  if (abs(vertical_translation) - edges_left*max_shift) > 3:
    #print("too large a vertical translation")
    #print("edges_left", edges_left)
    #print("number of edges", number_of_edges)
    #print("maxim", maxim)
    #print("max_shift", max_shift)
    #print("vertical_translation", vertical_translation)
    #print("foo > 3", abs(vertical_translation) - edges_left*max_shift)
    return True
  return False
```

A.2.4 *accept*(P, c)

```
def accept(P, c):
partial = c[0]
#print("checking to accept", partial.edges.data())
w = P[0]
k = P[1]
if len(partial.edges()) < w*k:
  return False
else:
  try:
    #print("trying to find a cycle")
    cycle = list(nx.find_cycle(partial))
  except:
    # no cycle is found
    #print("reject as no cycle was found")
    return False
  else:
```

```
    #print("we found a cycle")
    #print("cycle", cycle)
    #print("length of cycle", len(cycle))
    #print("3*w*k", 3*w*k)
    if len(cycle) == 3*w*k:
      # check the 'verticality'
      #print("checking the verticality")
      #print(partial.edges.data())
      vertical_translation = c[4]
      #print("vertical_translation", vertical_translation)
      if abs(vertical_translation) == 3*k:
        return True
      else:
        #print("rejected by verticality")
        return False
    else:
      return False
```

A.2.5 *first*(P, c)

```
def first(P, c):
partial = c[0]
partial_edges = partial.edges.data()
#print("partial_edges at start of first", partial_edges)
w = P[0]
k = P[1]
# if partial_edges is already full, there are no extensions
if len(list(partial_edges)) is 3*k*w:
  #print("partial edges is full")
  #print(partial_edges)
  return None
G = P[2]
tail_node = c[3]
second_to_last = c[5]
#print("tail_node", tail_node)
remaining = nx.MultiGraph()
remaining.add_nodes_from(G)
#print(list(G.edges.data(tail_node)))
#print(tail_node)
# add edges which are incident with tail_node
#print("add edges which are incident with tail_node")
for e in list(G.edges.data()):
  #print("e to be added", e)
  e_u = e[0]
  e_v = e[1]
  #print(e)
  #print(e_u, e_v)
  if (e_u == tail_node or e_v == tail_node):
    if (e_u == second_to_last or e_v == second_to_last):
      pass
    else:
      remaining.add_edge(e_u, e_v, key = e[2]['colour'],
colour = e[2]['colour'])
# remove the edges already in partial
# remove edges already in partial
for e in list(partial.edges.data()):
  try:
    remaining.remove_edge(e[0], e[1], key = e[2]['colour'])
  except:
    pass
#print("remove the edges that revisit already full nodes and remove them")
for n in list(partial.nodes()):
  #print("n", n)
  degree = partial.degree[n]
```

```
        #print("degree", degree)
        if degree == 2:
            #print("degree == 2, remove n")
            remaining.remove_node(n)
    # if remaining is empty, there are no more extensions
    remaining_edges = list(remaining.edges.data())
    #print("remaining_edges", remaining_edges)
    num_rem_edges = len(remaining_edges)
    if num_rem_edges is 0:
        #print("No more edges remaining")
        #print("partial_edges such that no more remaining edges", partial.edges.data())
        return None
    random_rem_edge = np.random.randint(num_rem_edges)
    candidate_edge = remaining_edges[random_rem_edge]
    #print("candidate_edge when first created in first", candidate_edge)
    ce_u = candidate_edge[0]
    ce_v = candidate_edge[1]
    ce_col = candidate_edge[2]['colour']
    remaining.remove_edge(ce_u, ce_v, key = ce_col)
    partial.add_edge(ce_u, ce_v, key = ce_col, colour = ce_col)
    old_tail_node = c[3]
    second_to_last = old_tail_node
    if ce_u == old_tail_node:
        new_tail_node = ce_v
    elif ce_v == old_tail_node:
        new_tail_node = ce_u
    else:
        raise Exception("Something has gone wrong, as neither ce_u nor ce_v",
    "is the tail_node, first function")
    s_to_l_translation = c[4]
    #print("s_to_l_translation", s_to_l_translation)
    shift_dict = P[6]
    shift = shift_dict[ce_col]
    #print("shift", shift)
    #print("tail_node", new_tail_node)
    #print("tail_node[0]", new_tail_node[0])
    if ((s_to_l_translation + shift)%(3*k) == new_tail_node[0]):
        new_vertical_translation = s_to_l_translation + shift
    elif ((s_to_l_translation - shift)%(3*k) == new_tail_node[0]):
        new_vertical_translation = s_to_l_translation - shift
    else:
        raise Exception("Something went wrong with the shift")
    #new_vertical_translation = s_to_l_translation + shift
    #print("new_vertical_translation", new_vertical_translation)
    #print("candidate_edge at end of first", candidate_edge)
    #candidate_list.append(partial.copy())
    new_c = (partial, remaining, candidate_edge, new_tail_node,
    new_vertical_translation, second_to_last, s_to_l_translation)
    return new_c
```

A.2.6 *next*(P, c)

```
def next_one(P, c):
partial = c[0]
remaining = c[1]
old_candidate_edge = c[2]
#print("partial edges", list(partial.edges.data()))
#print("old candidate edge", old_candidate_edge)
partial.remove_edge(old_candidate_edge[0],
old_candidate_edge[1],
key = old_candidate_edge[2]['colour'])
remaining_edges = list(remaining.edges.data())
num_rem_edges = len(remaining_edges)
if num_rem_edges is 0:
```

```
  return None
random_rem_edge = np.random.randint(num_rem_edges)
candidate_edge = remaining_edges[random_rem_edge]
ce_u = candidate_edge[0]
ce_v = candidate_edge[1]
ce_col = candidate_edge[2]['colour']
remaining.remove_edge(ce_u, ce_v, key = ce_col)
partial.add_edge(ce_u, ce_v, key = ce_col, colour = ce_col)
second_to_last = c[5]
if ce_u == second_to_last:
  tail_node = ce_v
elif ce_v == second_to_last:
  tail_node = ce_u
else:
  raise Exception("Something has gone wrong, as neither ce_u nor ce_v",
"is the tail_node, next_one function")
# need to pass through vertical_translation after computing it
s_to_l_translation = c[6]
#print("s_to_l_translation", s_to_l_translation)
shift_dict = P[6]
shift = shift_dict[ce_col]
#print("shift", shift)
#print("tail_node[0]", tail_node[0])
if ((s_to_l_translation + shift)%(3*k) == tail_node[0]):
  new_vertical_translation = s_to_l_translation + shift
elif ((s_to_l_translation - shift)%(3*k) == tail_node[0]):
  new_vertical_translation = s_to_l_translation - shift
else:
  raise Exception("Something went wrong with the shift")
#print("shift", shift)
#print("tail_node", tail_node)
#print("tail_node[0]", tail_node[0])
#print("second_to_last", second_to_last)
#print("second_to_last[0]", second_to_last[0])
new_c = (partial, remaining, candidate_edge, tail_node,
new_vertical_translation, second_to_last, s_to_l_translation)
return new_c
```

A.2.7 *output*(P, c)

```
def output(P, c, sol_list_old):
solution = c[0].copy()
sol_list.append(solution)
print("A SOLUTION", solution.edges.data())
#return sol_list
sys.exit()
```

A.2.8 The main method

```
# the width of the graph
w = 11

# the multiple height of the graph
k = 1

# The generating set
gen_set = [(1, 0, 'blue'), (2,0, 'red'), (0,1, 'green')]
first_edge = ((0,0), (1,0), 'blue')
shift_dict = {}
max_shift = 0
```

```
for g in gen_set:
  shift_dict[g[2]] = g[0]
  max_shift = max(max_shift, g[0])

# The graph of interest
G = nx.MultiGraph()

nodes = []

for x in range(3*k):
  for y in range(w):
    nodes.append((x,y))

G.add_nodes_from(nodes)

for (x,y) in nodes:
  for g in gen_set:
    u = (x + g[0])%(3*k)
    v = (y + g[1])%(w)
    G.add_edge((x,y), (u,v), key = g[2], colour = g[2])

P = (w, k, G, gen_set, first_edge, max_shift, shift_dict)

sol_list = []

candidate_list = []

bt(root(P), sol_list)
```

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